# Exercises of *The Mathematica GuideBook for Numerics*



## *CHAPTER* **1**

## Exercises

### 1.[L1] Logistic Map, Iterations with Noise

**a)** In connection with the study of bifurcations, it is interesting to investigate the result of iterating the mapping $f\colon z \to z^2 - c$. We call $z^*$ a period $n$ fixed point ([1881✱], [1339✱] and [1311✱]), provided the $n$th iteration of $f$ satisfies $f_n(z^*) = z^*$ where $f_n(z)$ is defined recursively by $f_n(z) = f(f_{n-1}(z))$, $f_1(z) = f(z)$.

Determine graphically the dependence of the (complex) period 1, 2, 3, and 4 fixed points on the real value $c$.

**b)** Visualize the real period 1, …, 7 fixed points of the map $x \to x^2 - c$ in the real $c$,$x$-plane [660✱], [1676✱], [932✱], [847✱].

**c)** Consider the sequence $\{x_0, x_1, x_2, \ldots\}$ where $x_n = f(x_{n-1})$ and $f(x) = \exp((x - 1/2)^2 / (3/10))$.

What happens with the sequences as a function of $x_0 \in \mathbb{R}^+$ when we disturb the iterations and use $x_n = f(x_{n-1} + \xi_n)$ ($\xi_n$ is a uniformly distributed random quantity from $(-1, 1)$) [1786✱], [1788✱], [1787✱]?

### 2.[L2] Analytic Functions with Finite Domains of Analyticity, $\cos_q(z)$ and $\sin_q(z)$

**a)** Visualize the function

$$f(z) = \sum_{k=1}^{\infty} z^{k!}.$$

Visualize how the function behaves when approaching the unit circle.

**b)** The $q$-trigonometric functions $\cos_q(z)$ and $\sin_q(z)$ are defined by the following series [1746✱], [1908✱], [1909✱], [1745✱], [568✱], [712✱], [1047✱], [238✱], [1180✱], [1640✱], [1391✱], [242✱], [1574✱], [1575✱]:

$$\cos_q(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{[2k]_q!} \frac{1}{q^k (q-1/q)^{2k}}$$

$$\sin_q(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{[2k+1]_q!} \frac{q^{k+1}}{(q-1/q)^{2k+1}}.$$

Here $[n]_q!$ is the $q$ factorial defined as

$$[n]_q ! = \prod_{k=1}^{n} \frac{q^k - q^{-k}}{q - q^{-1}}.$$

The $q$-trigonometric functions $\cos_q(z)$ and $\sin_q(z)$ obey the following identities similar to the identities for $\cos(z)$ and $\sin(z)$:

$$\frac{1}{z}(\sin_q(z) - \sin_q(q^{-2} z)) = \cos_q(z)$$
$$\frac{1}{z}(\cos_q(z) - \cos_q(q^{-2} z)) = -q^{-2} \cos_q(z)$$
$$\cos_q(z) \cos_q(q z) + q^{-1} \sin_q(z) \sin_q(q^{-1} z) = 1.$$

Implement the $q$-trigonometric functions $\cos_q(z)$ and $\sin_q(z)$ numerically for $|z| < \pi$, $|q| < \pi$. Check the above-mentioned identities for some randomly chosen numerical values of $z$ and $q$ to at least 100 digits. For a fixed $z$, visualize $\cos_q(z)$ as a function of $q$ in the complex $q$-plane.

### 3.[L2] `NDSolve` Fails, Integro-Differential Equation, Franel Identity, Bloch Particle in an Oscillating Field, Coupled and Accelerated Particles

**a)** Find an ordinary differential equation of first order that `NDSolve` is not able to solve.

**b)** Find a numerical approximation for the solution of the following integro-differential equation in the region $0 \le x \le 3\pi$:

$$y'(x) = \sin(x) + \frac{1}{\left(\arctan\left(1 + \left(\int_0^x y(t)\,dt\right)^2 + \int_0^x y(t)\,dt\right)\right)}.$$

**c)** Numerically determine the "exact value" of the constant $\alpha$ in the following identity:

$$\frac{1}{ab} \int_0^{ab} \left(\left\{\frac{x}{a}\right\} - \frac{1}{2}\right)\left(\left\{\frac{x}{b}\right\} - \frac{1}{2}\right) dx = \alpha \frac{\gcd(a, b)}{\operatorname{lcm}(a, b)}.$$

Here, $a$ and $b$ are positive integers and $\{x\}$ denotes the fractional part.

**d)** Calculate a reliable phase-space trajectory (in the $x(t)$,$x'(t)$-plane) for $0 \le t \le 200$ for the following differential equation: $x''(t) = -F + \sin(x(t) - \varepsilon \cos(\omega t))$, $x(0) = 0$, $p(0) = 10$ [715★]. Use the following parameter values $\varepsilon = 3/2$, $\omega = 5/3$, $F = 13/100$.

**e)** For a system of $n$ cyclically coupled, accelerated particles, described by the following system for their velocities $v_k(t)$

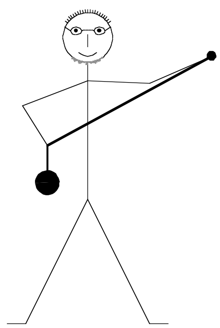
$$v_k'(t) = \frac{k-1}{n-1} + \varepsilon (\sin(v_{k+1}(t) - v_k(t)) + \sin(v_{k-1}(t) - v_k(t))), \; k = 1, \ldots, n,$$
$$v_0(t) = v_n(t), \; v_{n+1}(t) = v_1(t)$$

calculate the average velocity of the particles $\overline{v}_k$ ($\overline{v}_k \approx (v_k(T) - v_k(0))/T$ for large enough $T$) as a function of the coupling parameter $\varepsilon$ [455✱], [558✱].

### 4.$^{L2}$ Trick of James of Courtright, Hannay Angle, Harmonic Nonlinear Oscillator, Closed Orbits

**a)** Create a visualization of the Trick of James of Courtright with *Mathematica*. The trick is the following: Take a light object (e.g., a match box), and a heavier object (e.g., a set of keys), and tie them together with a string (of length about 1 m). Now, hang the string over a horizontal pencil so that the heavy object is as high as possible. Do this by holding the light object approximately at the height of the pencil ($\pm \approx 45^\circ$, measured against the horizontal). What happens if we now release the light object? Does the heavy object fall to the floor?

Here is the author carrying out this interesting experiment.



**b)** A bead is moving frictionless around an ellipse with half axes {1, 2}. Assume the ellipse is rotated with angular velocity $\omega$ around an axis that is perpendicular to the ellipse and goes through the focus. Calculate numerically to 20 digits how much the position of the bead of the rotated ellipse deviates from the position of a bead on a nonrotating ellipse in the limit $\omega \to 0$ ([194✱], [328✱], [1237✱]).

**c)** All generic solutions $\mathbf{M}(t)$ of the following nonlinear matrix oscillator are periodic (with period $T = 2\pi/\omega$) [332✱] (harmonic nonlinear oscillators [621✱], [333✱], [309✱]).

$$\mathbf{M}''(t) = 2\,\omega^2\,\mathbf{M}(t) + 3\,i\,\omega\,\mathbf{M}'(t) + c\,\mathbf{M}(t).\mathbf{M}(t).\mathbf{M}(t)$$

Here $\omega \in \mathbb{R}$, $c \in \mathbb{C}$, and $\mathbf{M}(t)$ is a $d \times d$ matrix. Visualize some solutions of this matrix differential equation.

**d)** The parametrized Hamiltonian $\mathcal{H} = \sigma\, p^{\alpha_1}\, r^{\beta_1} + r^{\beta_2}\, p^{\alpha_2}$ [549✱], [550✱] can be used to smoothly interpolate between the Hamiltonians of a harmonic oscillator $p^2 + r^2$ and the Kepler problem $p^2 - 1/r$ (here $p = |\boldsymbol{p}|$ and $r = |\boldsymbol{r}|$ and units were chosen to eliminate all prefactors). For the 2D case, construct a parametrized interpolating Hamiltonian and the corresponding equations of motion and make an animation how a family of orbits changes as one transits from the harmonic oscillator to the Kepler problem.

### 5.$^{L2}$ Eigenvalue Problems

Currently, `NDSolve` only solves initial-value problems. It does not work for eigenvalue problems, which arise frequently.

**a)** Using a shooting method (see, for instance, [1161✱]) (this means, try to find iteratively the initial conditions for the corresponding initial value problem) obtained by a combination of `NDSolve` and `FindRoot`, find the smallest eigenvalue of

$$-y''(x) + x^4\, y(x) = \lambda\, y(x)$$

(the quantum-mechanical steady state energy of an anharmonic oscillator). (For some more advanced methods to calculate energy values of anharmonic oscillators, see [1875✱], [1874✱], and [1193✱].)

**b)** Discretizing the eigenvalue problem [1675✱]

$$-y''(x) + f(x)\, y(x) - \lambda\, y(x) = 0,\; y(x_0) = y(x_{n+1}) = 0$$

using $n+2$ points $x_0, x_1, \ldots, x_n, x_{n+1}$, we get the following system of equations:

$$\begin{aligned} -\frac{1}{\Delta^2}\,(y_{i-1} - 2\, y_i + y_{i+1}) + f(x_i)\, y_i - \lambda\, y_i &= 0 \\ \Delta &= \frac{x_{n+1} - x_0}{n+1} \\ x_i &= x_0 + i\,\Delta,\; i = 1, \ldots, n+1 \\ y_i &= y(x_i),\; y_0 = y_{n+1} = 0. \end{aligned}$$

This linear system in the $y_i$ is solvable, provided that the coefficient determinant $D_n$ of the $y_i$ vanishes: $D_n = 0$. The direct computation of this determinant from the coefficient matrix is inefficient for large values of $n$, because the number of nonvanishing elements in the matrix grows like $n$, but the total number of elements grows like $n^2$. The determinant of this tridiagonal symmetric system can be computed easily by recursion [1832✱], [1215✱], [1061✱], [874✱], [556✱], [1935✱], [1865✱], [1286✱], [557✱] (for simplicity, we multiply by $\Delta^2$):

$$\begin{aligned} D_1 &= w_1 \\ D_2 &= w_1\, w_2 - 1 \\ D_i &= w_i\, D_{i-1} - D_{i-2} \\ w_i &= 2 + (f(x_i) - \lambda)\,\Delta^2. \end{aligned}$$

Implement the approximate computation of the eigenvalues of the above differential operator for arbitrary, nonsingular $f$, and moderate $n$ by determining the zeros of $D_n$. Use the program to find the eigenvalues for some examples.

**c)** The eigenvalue problem

$$-y''(x) + V(x)\, y(x) = \lambda\, y(x), \quad y(-\infty) = y(\infty) = 0$$

for $V(x) = x^2 + \sum_{k=0}^{o} c_k\, x^k$ can be solved by solving the finite eigenvalue problem for the matrix $(h_{i,j})_{i,j=-n,\ldots,n}$ [897✱]:

$$h_{i,j} = \delta_{i,j-2}\,(1-\omega_0^2)\,\frac{\sqrt{j\,(j-1)}}{2\,\omega_0} + \delta_{i,j+2}\,(1-\omega_0^2)\,\frac{\sqrt{(j+1)\,(j+2)}}{2\,\omega_0} + \delta_{i,j}\left(\frac{j\,(\omega_0^2+1)}{\omega_0} + \frac{\omega_0^2+1}{2\,\omega_0}\right) + \sum_{m=0}^{o}\sum_{k=0}^{\lfloor m/2\rfloor}\sum_{n=0}^{m-2k}\delta_{i,j-2k+m-2n}\,\frac{c_m\, m!\,\sqrt{j!\,(j-2k+m-2n)!}}{(2\,\omega_0)^{m/2}\, 2^k\, k!\, n!\,(j-n)!\,(m-n-2k)!}$$

Here $\omega_0$ is implicitly defined through (optimized harmonic oscillator basis [903✱])

$$\sum_{\substack{m=2 \\ \Delta m=2}}^{o} \frac{c_m\, m!}{2^{m-1}\,\omega_0^{m/2+1}\,(m/2-1)!} = 1 - \frac{1}{\omega_0^2}.$$

Use these formulas to calculate the two lowest eigenvalues for $V(x) = x^4 - 4\,x^2$ to 10 correct digits.

**d)** In a $2\,n+1$-dimensional representation of the Weyl system (Schwinger representation [1876✱], [655✱], [1579✱], [773✱], [1877✱], [479✱]) where differentiation operators are represented as $\mathcal{F}_{2n+1}^{-1}\, X\, \mathcal{F}_{2n+1}$ ($X$ being the multiplication operator and $\mathcal{F}_{2n+1}$ the discrete Fourier transform of order $2\,n+1$), the approximate eigenvalues of the differential operator $\frac{\partial^2}{\partial x^2} + V(x)$ ($V(x)$ being sufficiently smooth) are given by the eigenvalues of the matrix $\mathcal{M} = (m_{k,l})_{k,l=-n,\ldots,n}$ [514✱], [515✱], [1637✱]:

$$m_{k,l} = \begin{cases} \frac{(-1)^{l-k}}{\nu}\,\pi\cot\left(\frac{(l-k)\pi}{\nu}\right)\csc\left(\frac{(l-k)\pi}{\nu}\right) & k \neq l \\ \frac{(\nu^2-1)}{6\nu}\,\pi + V\left(k\sqrt{\frac{2\pi}{2\nu+1}}\right) & k = l. \end{cases}$$

For $V(x) = x^2$ and $n = 10, 50$ calculate the first few eigenvalues and compare with the precision of the result with the case from the last subexercise where the differentiation operator was represented using a finite difference approach. Which $n$ is needed to obtain 20 correct digits for the ground-state of the quartic oscillator $V(x) = x^4$?

**e)** For the time-independent Schrödinger equation $-\psi_j''(\alpha; x) + V(\alpha; x)\,\psi_j(\alpha; x) = \varepsilon_j(\alpha)\,\psi_j(\alpha; x)$ with the piecewise constant "Möbius potential"

$$V(\alpha; x) = \alpha \sum_{k=0}^{n} (2\,\mu(k) + 1)\,\theta(x - x_k)\,\theta(x_{k+1} - x),$$

where $\mu(k)$ is the value of the Möbius function $\mu(k)$ (in *Mathematica* `MoebiusMu[k]`) at the integer $k$, determine the dependence of the first ten eigenvalues in the range $0 \le \alpha \le 2$. $\psi_\alpha(x)$ fulfills Dirichlet boundary conditions $\psi_\alpha(\alpha; 0) = \psi_\alpha(\alpha; n+1) = 0$, and let $n = 50$. Determine for which $\alpha^*$ the lowest eigenvalue $\varepsilon_1$ coincides with $\alpha$; this means $\varepsilon_1(\alpha^*) = \alpha^*$. Visualize the dependence of the $\psi_j(\alpha; x)$ on $\alpha$. (For the physics of 1D nonperiodic, piecewise constant potentials, see [590✱], [1250✱], [1187✱], [365✱], [35✱], [649✱], [1058✱], [845✱], [1592✱], [1031✱], [307✱], [74✱], [1380✱], [904✱], [1283✱], [1748✱], [667✱], [1382✱], [486✱], [369✱], and [1769✱].)

**f)** Investigate the symmetric solutions of $-\psi_\varepsilon''(x) + V(x)\,\psi_\varepsilon(x) = \varepsilon\,\psi_\varepsilon(x)$ for the potential

$$V(x) = -\theta(|x|)\,\theta\left(\frac{\pi}{2} - |x|\right) - 4\,\theta\left(|x| - \frac{\pi}{2}\right)\theta(\pi - |x|) - \sum_{k=3}^{\infty} k^2\,\theta(|x| - (k-2)\,\pi)\,\theta((k-1)\,\pi - |x|)\,.$$

Visualize the solutions for some $\varepsilon$. For which $\varepsilon$ will the solutions be square integrable?

### 6.[L2] Wynn's ϵ-Algorithm, Aitken Algorithm, Numerical Regularization

To illustrate how to find the infinite limiting value of a finite sequence, we consider Wynn's ϵ-algorithm and the Aitken algorithm (these are very near relatives of each other). The idea is to successively reduce the length of the sequence in such a way that the resulting terms approach the infinite limiting value [366✱], [1902✱], [1903✱].

**a)** We begin with a sequence $\epsilon_0$, whose $n$th element is $\epsilon_0^{(n)}$. We define $\epsilon_{-1}^{(n)} = 0$ for all $n$. The (shortened) sequence at the $k$th step is in Wynn's ϵ-algorithm obtained by (see [1949✱], [1923✱], [749✱], [1658✱], [477✱], [1901✱], [911✱], [288✱] and [755✱] for details):

$$\epsilon_k^{(n)} = \epsilon_{k-2}^{(n+1)} + \frac{1}{\epsilon_{k-1}^{(n+1)} - \epsilon_{k-1}^{(n)}}.$$

(We suppose in the latter formula that this operation is well defined, i.e., there is no division by zero.) The results make sense only for even $k$. Implement this form of Wynn's ϵ-algorithm, and apply it to some test sequences.

**b)** Aitken transformation [30✱], [287✱], [208✱], and [1593✱] can be written in the following form:

$$\epsilon_k^{(1)} = \epsilon_0^{(k)}$$

$$\epsilon_k^{(n)} = \frac{\epsilon_{k-1}^{(n+1)}\,\epsilon_{k-1}^{(n-1)} - \left(\epsilon_{k-1}^{(n)}\right)^2}{\epsilon_{k-1}^{(n+1)} + \epsilon_{k-1}^{(n-1)} - 2\,\epsilon_{k-1}^{(n)}} \qquad n \ge 2.$$

Here again $\epsilon_k^{(n)}$ is the $n$th element of the $k$th transformation of the given sequence $\epsilon_0$ with terms $\epsilon_0^{(k)}$. Write a functional program that carries out the Aitken's transformation as long as possible. Again, test the transformation for some examples.

**c)** Consider the following divergent sum [173✱]:

$$\lim_{n\to\infty}\sum_{k=1}^{n}(k^2+k)\ln(k).$$

Use purely numerical techniques to calculate a five-digit approximation of Hadamard-regularized version of this sum (meaning that all divergencies of the form $n^k \ln^l(n)$, $k$, $l \in \mathbb{N}$ are subtracted).

### 7.[L2] Scherk's Fifth Surface, Clebsch's Surface, Holed Dodecahedron, Smoothed Dodecahedron

**a)** Plot the surface defined implicitly by $\sin(z) = \sinh(x)\sinh(y)$ (Scherk's fifth surface) over a large domain of $z$-values. What peculiarities does it exhibit? Do *not* just use

```
ContourPlot3D[Sin[z] - Sinh[x] Sinh[y],{x, -3, 3}, {y, -3, 3}, {z, 0, 4Pi},
              Contours -> {0.0}, MaxRecursion -> 1,
              PlotPoints -> {{6, 5}, {6, 5}, {12, 5}}].
```

**b)** Make a pretty picture of the surface defined by the implicit equation

$$32 - 216\,x^2 + 648\,x^2\,y - 216\,y^2 - 216\,y^3 - 150\,z + 216\,x^2\,z + 216\,y^2\,z + 231\,z^2 - 113\,z^3 = 0$$

using `ContourPlot3D` and using the symmetry of the surface. Try to make the picture of this surface without using `ContourPlot3D`.

**c)** Given the following definitions (after loading the package ``Graphics`Polyhedra` ``):

```
polyWithPlatoSymmetry[plato_, {x_, y_, z_}, n_] :=
Plus @@ (({x, y, z}.#)^n & /@ ((Plus @@ #/Length[#]& /@
(First /@ (Polyhedron[plato][[1]]))))) - 1

dodePoly = Chop[polyWithPlatoSymmetry[Dodecahedron, {x, y, z}, 14]]

cylinderAxisDirections = #/Sqrt[#.#]& /@ Chop[Apply[Plus,
  First /@ Take[Polyhedron[Dodecahedron][[1]], 6], {1}]/5]

cylinderPoly = Times @@ (x^2 + y^2 + z^2 - ({x, y, z}.#)^2 - 0.01 & /@
                                                    cylinderAxisDirections);

thePoly[{x_, y_, z_}] = dodePoly cylinderPoly + 3 10^-2;
```

What is the form of the surface implicitly defined by `thePoly[{x, y, z}] == 0`? Make a picture of the inner part of this surface without using `ContourPlot3D`.

**d)** Construct a thickened wireframe version of a dodecahedron made from triangles. The polygonal mesh forming the thickened wireframe should have at least $10^4$ triangles. Smooth the thickened wireframe by replacing the coordinates of each point with the average of all neighboring points. Iterate this smoothing procedure 1000 times.

### 8.[L1] A Convergent Sequence for $\pi$, Contracting Interval Map

**a)** Investigate the convergence of the sequence $1/\alpha_n$ to $\pi$, where

$$
\begin{aligned}
y_{n+1} &= \frac{1-\sqrt[4]{1-y_n^4}}{1+\sqrt[4]{1-y_n^4}} \\
\alpha_{n+1} &= (1+y^{n+1})^4\,\alpha_n - 2^{2n+3}\,y_{n+1}\,(1+y_{n+1}+y_{n+1}^2) \\
y_0 &= \sqrt{2}-1 \\
\alpha_0 &= 6-4\sqrt{2}.
\end{aligned}
$$

Do the same for the following sequence.

$$
\begin{aligned}
y_{n+1} &= \frac{25\,v_n^2}{(v_n^2+u_n+v_n)^2\,y_n} \\
u_n &= \frac{5}{y_n}-1 \\
v_n &= \sqrt[5]{\frac{u_n}{2}\left((u_n-1)^2+7+\sqrt{((u_n-1)^2+7)^2-4\,u_n^3}\right)} \\
\alpha_{n+1} &= y_n^2\,\alpha_n - \frac{5^n}{2}\left(y_n^2-5-2\sqrt{y_n(y_n^2-2\,y_n+5)}\right) \\
y_0 &= 5\left(\sqrt{5}-2\right) \\
\alpha_0 &= \frac{1}{2}.
\end{aligned}
$$

For derivations of these iterations, see [115✱], [249✱], [870✱].

**b)** Consider the map $x \to \{\alpha\,x+\beta\}$ with $0<\alpha,\ \beta<1$ and $\{x\}$ denoting the fractional part of $x$ [315✱]. Starting with the interval [0, 1], visualize the repeated application of the map for various $\alpha$, $\beta$ .

### 9.[L1] Standard Map, Stochastic Webs, Iterated Cubics, Hénon Map, Triangle Map

**a)** The so-called standard mapping is an iterative mapping of $(0,1)\times(0,1)$ into itself. It is defined by

$$
\begin{aligned}
x_{i+1} &= \left(x_i+y_i-\frac{K}{2\pi}\sin(2\pi\,x_i)\right)\bmod 1 \\
y_{i+1} &= \left(y_i-\frac{K}{2\pi}\sin(2\pi\,x_i)\right)\bmod 1.
\end{aligned}
$$

Depending on the choice of $K$ and the starting point $\{x_0,\ y_0\}$, we get very different “movements” of the points $\{x_i,\ y_i\}$. For randomly chosen starting points, examine the movement for $0\le K\le 3$.

**b)** A related mapping is

$$
\begin{aligned}
x_{i+1} &= \sin(\alpha)\,y_i+\cos(\alpha)\,(x_i+K\sin(2\pi\,y_i)) \\
y_{i+1} &= \cos(\alpha)\,y_i+\sin(\alpha)\,(x_i+K\sin(2\pi\,y_i)).
\end{aligned}
$$

For various choices of $\alpha$ (e.g., rational approximations of $\pi$ with small denominators) and randomly chosen starting points, we get very interesting patterns, called stochastic webs [1973✱], [80✱], [22✱], [1972✱], [1967✱]. Examine some examples. In addition, make an animation that shows how an initially localized distribution diffuses into the observed structures from iterating points [354✱].

**c)** Given the following two maps [244✱]

$$\text{map 1}: \begin{array}{ll} p_{i+1} & = (p_i + K \sin(\theta_i)) \bmod 2\pi \\ \theta_{i+1} & = \theta_i + p_{i+1} \end{array}$$

$$\text{map 2}: \begin{array}{ll} p_{i+1} & = (p_i + K \sin(\theta_i) \operatorname{sign}(\cos(\theta_i))) \bmod 2\pi \\ \theta_{i+1} & = \theta_i + p_{i+1.} \end{array}$$

Visualize how the two mappings transform into each other by varying the parameter $t$ $(0 \leq t \leq 1)$ in

$$\begin{array}{ll} p_{i+1} & = (p_i + K((1-t)\sin(\theta_i) + t\sin(\theta_i)\operatorname{sign}(\cos(\theta_i)))) \bmod 2\pi \\ \theta_{i+1} & = \theta_i + p_{i+1}. \end{array}$$

Choose $K$ small, say, $K \approx 1/100$.

**d)** The map [1641✱]

$$\begin{array}{ll} x_{i+1} & = \sin(\omega)(p_i - \mu \operatorname{sgn}(x_i)) + \cos(\omega) x_i \\ y_{i+1} & = \cos(\omega)(p_i - \mu \operatorname{sgn}(x_i)) - \sin(\omega) x_i \end{array}$$

exhibits a wide variety of possible phase space patterns. Find at least 10 pairs of values for the parameters $\{\omega, \mu\}$ that show "different" patterns.

**e)** The forces logistic map [461✱]

$$\begin{array}{ll} x_{i+1} & = \alpha x_i (1 - x_i) + \varepsilon \sin(2\pi \vartheta_i) \\ \vartheta_{i+1} & = (\vartheta_i + \omega) \bmod 1 \end{array}$$

with parameters $\omega$, $\alpha$, and $\varepsilon$ exhibits a variety of possible patterns. Visualize some of the resulting patterns in the $\vartheta$,$x$-plane.

**f)** The following implementation of the so-called web map [1510✱], [1974✱], [1975✱]

$$\begin{array}{l} x_{n+1} = y_n \\ y_{n+1} = -x_n - \kappa \sin(y_n) \end{array}$$

```
iList[κ_, {x0_, y0_}, n_] := First /@
      NestList[{#[[2]], -#[[1]] - κ Sin[#[[2]]]}&, {x0, y0}, n]

ListPlot[iList[2.178805237252476,
               {-2.9085181016961186, 0.8317263342785042}, 20000],
         Frame -> True, Axes -> False];
```

shows four nearly parallel "lines" for $5500 \lesssim n \lesssim 8500$. Are these nearly parallel "lines" numerical artifacts or are they real? If they are real, find other $\kappa$, $\{x_0, y_0\}$-values so that `ListPlot[iList[`$\kappa$`, `$\{x_0, y_0\}$`, `$n$`]]` shows regions of four parallel "lines".

**g)** The points in $\mathbb{R}^2$ obtained from the iteration of the bivariate polynomial map

$$\{x, y\} \longrightarrow \left\{ \sum_{k,l=0}^{n} c_{k,l}^{(x)} x^k y^l, \sum_{k,l=0}^{n} c_{k,l}^{(y)} x^k y^l \right\}$$

lead, for appropriately chosen $c_{k,l}^{(x,y)}$, to strange attractors [404✱], [1710✱], [609✱], [1709✱], [694✱]. For $n = 3, 4, 5$ and $-1 \leq c_{k,l}^{(x,y)} \leq 1$, find such strange attractors by carrying out automatic searches. For randomly chosen $c_{k,l}^{(x,y)}$, how frequently does one encounter a strange attractor?

**h)** Consider the following input (the map comes from [799✱]).

```
step[x_, {P_, e_}] :=
Module[{p = If[Random[] < P, 1/2 + e, 1 - (1/2 + e)],
        η = If[Random[] < P, 1, 0]},
       η (1 - p)/(1 - (1 - p) x) + (1 - η) (1 - p)/(1 - p x)]

SeedRandom[82313041463260884204]
NestList[step[#, {0.47629739820406214, 0.40133038775291635}]&,
          0.01011450108382423, 10^4];
```

Is the result mathematically correct? Write a compiled function that correctly carries out this iteration.

**i)** Given two lists of length $l$ of real numbers $\{u_i\}$, $\{v_i\}$, $i = 1, \ldots, l$, consider the mapping [231✱], [377✱]

$$u_i^{(n+1)} = f_r(u_i^{(n)}) + \frac{\varepsilon}{2}\left(f_r(u_{i-1}^{(n)}) - 2 f_r(u_i^{(n)}) + f_r(u_{i+1}^{(n)})\right) + v_i^{(n)}$$
$$v_i^{(n+1)} = b(u_i^{(n+1)} - u_i^{(n)}).$$

Here the function $f$ is defined as

$$f_r(x) = \begin{cases} r\,x & \text{if } x \le \frac{1}{2} \\ r(1-x) & \text{if } \frac{1}{2} \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$$

and $r$, $\varepsilon$, and $b$ are real parameters. Find triples of values $\{r, \varepsilon, b\}$ such that the data $u_i^{(n)}$ show qualitatively different behavior over the discrete $i$,$n$-plane. Use periodic boundary conditions $u_i^{(n)} = u_{i+l}^{(n)}$, $v_i^{(n)} = v_{i+l}^{(n)}$ and random initial $u_i^{(0)}$ and $v_i^{(0)}$.

**j)** The iterations of the Hénon map [827✱], [1563✱]

$$x_{i+1} = A + B\,y_i - x_i^2$$
$$y_{i+1} = x_i$$

exhibit a variety of possible patterns. Show various “types” of patterns that arise from starting with $\{x_0, y_0\}$ located on a circle. Conduct an automated search for “interesting” sets of parameters $A$ and $B$ and initial circles.

**k)** The map $z \to z^2 - (1 - i\,\lambda)\,\bar{z}$, where $\lambda$ is the largest positive root of $3\,\lambda^2\,(9\,\lambda^4 + 3\,\lambda^2 + 11) = 77$, has in the complex plane the line $\{-(3\,\lambda^2 + 1)/4 + i\,\lambda/2, (9\,\lambda^4 + 6\,\lambda^2 + 5)/16 + i\,\lambda/2\}$ and its two images under counterclockwise rotation by $2\,\pi/3$ and $4\,\pi/3$ as attractors [38✱]. The basins of attraction of these three lines are intermingled. Sketch these intermingled basins. Carry out the calculations using machine and high-precision arithmetic.

**l)** Consider the map [34✱]

$$x_m(s) = \beta_0 + \sum_{j=1}^{n} \beta_j \tanh\left(s\,\omega_j + \sum_{k=1}^{d} \omega_{j\,k}\,x_{m-k}(s)\right).$$

For each map, $\beta_0$ is a uniformly distributed variable from [0, 1], the $\beta_j$ are rescaled (to $\sum_{j=1}^{n} \beta_j^2 = n$) uniformly distributed variable from [0, 1], and the $\omega_j$ and $\omega_{j\,k}$ are independently and identically distributed Gaussian random variables with mean 0 and variance 1. The $d$ initial values $x_1, x_2, \ldots, x_d$ are assumed to be uniformly distributed variable from $[-1, 1]$.

For $n \approx d \approx 2^6$ and a concrete realization of the random variables, make an animation showing how the set of points $\{x_{m-1}(s), x_m(s)\}$ evolve as a function of the real parameter $s$.

## 10.[L2] Movement in a 2D Periodic and Egg Crate Potential, Pearcey Integral, Subdivision of Surface, Construction of Part of Surface, Electric Field Visualizations

**a)** Suppose an electrically charged particle moves in a periodic potential of the form

$$V(x, y) = V_m \cos(x) + V_m \cos(y).$$

In addition, suppose we impose a static homogeneous electric field in the $x$-direction, and a static homogeneous magnetic field in the $z$-direction. Compute some resulting paths of the particle, and plot them.

**b)** Suppose a particle moves in a periodic potential of the form [677✱], [1010✱], [1975✱]

$$V(x, y) = \frac{1}{2}(5 + 3(\cos(x) + \cos(y)) + \cos(x)\cos(y)).$$

Show some "qualitatively different" trajectories. How sensitive are these trajectories qualitatively and quantitatively depending on the initial conditions?

**c)** Make a contour plot of the absolute value and the phase of the following function $P(x, y)$ (Pearcey integral)

$$P(x, y) = \int_{-\infty}^{+\infty} e^{i(u^4 + x u^2 + y u)}\, du$$

for real values of $\{x, y\}$ around $\{0, 0\}$. $P(x, y)$ can be effectively computed by solving the following two ordinary differential equations:

$$\begin{aligned}
4\frac{\partial^2}{\partial x^2}P(x,0) + 2\,i\,x\frac{\partial}{\partial x}P(x,0) + i\,P(x,0) &= 0\\
P(x,0)\big|_{x=0} &= \frac{(-1)^{(1/8)}}{2}\,\Gamma(1/4)\\
\frac{\partial}{\partial x}P(x,0)\Big|_{x=0} &= \frac{i\,(-1)^{(3/8)}}{2}\,\Gamma(3/4)
\end{aligned}$$

$$\begin{aligned}
4\frac{\partial^3}{\partial y^3}P(x,y) - 2\,x\frac{\partial^2}{\partial y^2}P(x,y) - i\,y\,P(x,y) &= 0\\
P(x,y)\big|_{y=0} &= P(x,0)\\
\frac{\partial}{\partial y}P(x,y)\Big|_{y=0} &= 0\\
\frac{\partial^2}{\partial y^2}P(x,y)\Big|_{y=0} &= i\frac{\partial}{\partial x}P(x,0).
\end{aligned}$$

(see [1290✱], [1432✱], [1372✱], [195✱], [433✱], [431✱], [1067✱], [432✱], [1716✱], [939✱], [193✱], [434✱], and [1420✱] for details).

**d)** Given the parametrically defined surface (the Klein bottle discussed in Chapter 2 of the Graphics volume [1807✱])

$$\begin{aligned}
x(s,t) &= (2 + \cos(s/2)\sin(t) - \sin(s/2)\sin(2t))\cos(s)\\
y(s,t) &= (2 + \cos(s/2)\sin(t) - \sin(s/2)\sin(2t))\sin(s)\\
z(s,t) &= \sin(s/2)\sin(t) + \cos(s/2)\sin(2t)
\end{aligned}$$

divide the $s$,$t$-plane in $n_s$,$n_t$ parts, so that the corresponding pieces of the surface all have the same area.

**e)** Make a picture of the part of the surface that is visible in the following picture (do not use the `PlotRange -> …` restriction).

```
r[φ_, z_] = (1 + 2 Sin[z/Pi]^2 Cos[2φ]^6);
```

```
ParametricPlot3D[Evaluate[{r[φ, z] Cos[φ], r[φ, z] Sin[φ], z,
{EdgeForm[Thickness[0.002]],
 SurfaceColor[RGBColor[0, 1, 0], RGBColor[1, 1, 0], 2.6]}}],
                {φ, 0, 2Pi}, {z, 0, Pi},
                PlotPoints -> {36, 46}, BoxRatios -> {1, 1, 2},
                PlotRange -> {{-5/4, 5/4}, {-5/4, 5/4}, {0, Pi}},
                Boxed -> False, Axes -> False];
```

**f)** Given a random trigonometric function (such as

```
Sum[Random[] Cos[i x + 2 Pi Random[]] Cos[j y + 2 Pi Random[]],
    {i, 0, n}, {j, 0, n}])
```

calculate curves that start and end at extremal points and follow the gradient.

**g)** Make an animation that shows the equipotential lines of two superimposed finite square (hexagonal) grids made from line segments. Let the angle the two grids form be the animation parameter. Try for an efficient implementation.

### 11.[L2] A Ruler on the Fingers, Trajectories in Highly Oscillating Potential, Branched Flows

**a)** Describe the following experiment via a numerical solution of Newton's equation of motion. Hold a ruler horizontally on your index fingers (about 20 to 40 cm apart). Now move the fingers horizontally toward each other at a nearly uniform velocity. This causes the ruler to move alternately left and right. In computing the solution of this problem, be especially careful where the direction of movement changes. What would be the result if there were no static friction?

**b)** Calculate and visualize 100 trajectories (with randomly chosen initial conditions) of a particle moving in the 2D potential $V(x, y) = \sin(\cot(y - x^2) + \tan(x + y))$.

**c)** Visualize the motion of many particles moving under the simultaneous influence of a smooth, random potential and a monotonic potential.

### 12.[L1] Maxwell's Line, Quartic Determinant

**a)** Plot the isothermals of a van der Waal's gas on a $p,V$-diagram (compute Maxwell's line). For details on the construction of Maxwell's line, see, for example, [859✱], [1427✱], [303✱], [1080✱], [1885✱], [79✱], and [647✱]. For a general treatment of van der Waal's gas, see [1827✱], [934✱], [822✱], [823✱], and [4✱].

**b)** Calculate realizations for $a_{i,j}$, $b_{i,j}$, and $c_{i,j}$ such that the following determinantal identity holds [540✱]:

$$\left|\begin{pmatrix} 0 & a_{1,2}\,x + b_{1,2}\,y + c_{1,2}\,z & a_{1,3}\,x + b_{1,3}\,y + c_{1,3}\,z & a_{1,4}\,x + b_{1,4}\,y + c_{1,4}\,z \\ a_{1,2}\,x + b_{1,2}\,y + c_{1,2}\,z & 0 & a_{2,3}\,x + b_{2,3}\,y + c_{2,3}\,z & a_{2,4}\,x + b_{2,4}\,y + c_{2,4}\,z \\ a_{1,3}\,x + b_{1,3}\,y + c_{1,3}\,z & a_{2,3}\,x + b_{2,3}\,y + c_{2,3}\,z & 0 & a_{3,4}\,x + b_{3,4}\,y + c_{3,4}\,z \\ a_{1,4}\,x + b_{1,4}\,y + c_{1,4}\,z & a_{2,4}\,x + b_{2,4}\,y + c_{2,4}\,z & a_{3,4}\,x + b_{3,4}\,y + c_{3,4}\,z & 0 \end{pmatrix}\right| = 2\,i\,(x^4 + y^4 + z^4).$$

It is known that some of the $a_{i,j}$, $b_{i,j}$, and $c_{i,j}$ are $\pm 1$.

### 13.[L2] Smoothing Functions, Secant Method Iterations, Unit Sphere Inside a Unit Cube

**a)** Plot the function

$$h_\epsilon(x) = \int_{-\infty}^{\infty} g_\epsilon(y)\, f(x-y)\, dy$$

with

$$g_\epsilon(x) = \begin{cases} \exp(1/(x^2-\epsilon^2)) \Big/ \int_{-\epsilon}^{\epsilon} \exp(1/(x^2-\epsilon^2))\, dx & -\epsilon \le x \le \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \sin(x) & 0 \le x \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

for $\epsilon = 4, 2, 1, 1/2$, $0 \le x \le 2\pi$. Be careful that no error messages are generated during the computation.

**b)** How does the pattern in the following graphic arise?

```
DensityPlot[x /. FindRoot[Cos[x] - x, {x, x1, x2}, Method -> Secant,
            MaxIterations -> 30], {x1, -10, 10}, {x2, -10, 10},
            Compiled -> False, PlotPoints -> 200,
            Mesh -> False, ColorFunction -> (Hue[0.8 #]&)]
```

**c)** The part $a_d(\rho)$ of a $d$D sphere of radius $\rho$ that is inside a $d$D unit cube centered at the origin (and, without loss of generality, oriented along the coordinate axes) is given by

$$a_d(\rho) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \delta\Big(\rho - \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2}\Big)\, dx_1\, dx_2 \ldots dx_d.$$

Integrating the Dirac delta function gives the following recursion relation for the area [1035✱]

$$a_d(\rho) = 2\,\rho \int_{\max\left(0,(\rho^2-1/4)^{1/2}\right)}^{\min(\rho,(d-1)^{1/2}/2)} \frac{a_{d-1}(r)}{\sqrt{\rho^2-r^2}}\, dr.$$

Starting with $a_1(\rho) = 2\,\theta(\rho)\,\theta(1/2-\rho)$, calculate numerically and visualize $a_d(\rho)$ for $1 \le d \le 25$.

### 14.[L1] Computation of Determinants, Numerical Integration, Binary Trees, Matrix Eigenvalues

**a)** Implement the computation of determinants using Laplace expansion by minors [1200✱], and using this implementation, find the determinant of

```
Array[Which[#2 - 2 <= #1 <= #2 + 2, N[#1 + #2], True, 0]&, {5, 5}].
```

**b)** Decide purely numerically if the value of the following integral is $1/2$ [253✱], [254✱], [257✱], [281✱].

$$\int_0^\infty \left(\prod_{k=0}^{7} \frac{(2k+1)}{x} \sin\!\left(\frac{(2k+1)}{x}\right)\right) dx \overset{?}{=} \frac{1}{2}.$$

**c)** Confirm to at least 10 digits that the integral $\mathcal{I} = \int_0^\infty x \ln(x)\,(1 - y(x)^2)\,dx$ has the value $\mathcal{I} = 1/4 + 7/12 \ln(2) - 3 \ln(\mathcal{G})$ where $\mathcal{G}$ is the Glaisher constant (in *Mathematica* `Glaisher`) [137✱], [1945✱]. Here $y(x)$ is the solution of the nonlinear differential equation (Painlevé 3)

$$y''(x) = y(x)^3 + \frac{y'(x)^2}{y(x)} - \frac{y'(x)}{x} - \frac{1}{y(x)}$$

obeying the boundary conditions $y(x) \underset{x\to 0}{\approx} -x\,(\ln(x/4) + \gamma)$ and $y(x) \underset{x\to\infty}{\approx} 1 - (\pi\, x)^{-1/2} \exp(-2\,x)$.

**d)** Generate "random binary trees" (meaning start with a binary tree of fixed depth and then randomly delete branches). Generate the corresponding adjacency matrices (containing a 1 in position $(i, j)$ if the node $i$ of the random binary tree is connected to the node $j$ and 0 else). Analyze the eigenvalue distribution of the adjacency matrices for 100 random binary trees that were derived from full binary trees of depth eight.

**e)** Carry out some numerical experiments to conjecture an approximate formula describing the value of the $k$th eigenvalue (sorted by decreasing magnitude) $\lambda_k^{(n)}$ of an $n \times n$ left triangular matrix with elements 1. Consider the case $n \gg 1$.

**f)** The expectation value of the $k$th power of the determinant of a $n \times n$ matrix with entries $\pm 1$ is [161✱]

$$\overline{\det\left((\pm 1)_{\substack{i=1,\ldots,n\\ j=1,\ldots,n}}\right)} = i^{k\,n} \left(\left(\det\left(\left(\frac{\partial}{\partial z_{i,j}}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,n}}\right)\right)^k \prod_{i=1}^{n} \prod_{j=1}^{n} \cos(z_{i,j})\right)\Bigg|_{z_{i,j}=0}.$$

where $(\det|\ldots|)^k$ represents the $k$-fold application of the differential operator obtained from taking the determinant and the overbar denotes averaging. For $1 \le k, n \le 4$, calculate the expectation values and compare them with the average of $10^6$ random realizations of the matrices.

### 15.[L1] Root Pictures, Weierstrass Iterations, Taylor Series Remainders

**a)** In analogy with the Goffinet picture of Chapter 1 of the Graphics volume [1807✱], visualize all of the following sums:

$$\sum_{i=1}^{m} j_i\, x_i \quad i = 1, \ldots\ n,$$

where the $x_i$ are the zeros of a given polynomial of $n$th degree and the $j_i$ are independently taken from the possible values $0 \le j_i \le n$.

**b)** Construct $n$ random polynomials $p_j(z)$ of degree $j$. Then, starting from a polynomial $q(z)$ of degree $n$, make the following substitution in $q(z)$: $z^j \to p_j(z)$. Iterate this process, and make a picture of the zeros of these iterated polynomials. (For details on this process with $p_j(z)$ as the orthogonal polynomials, see [881✱].)

**c)** The polynomial $x^2 + x + 1 = 0$ has two roots $x_{1,2}$. Adding a term $\varepsilon\, x^3$ to this polynomial results in a third root $x_3$. For small $\varepsilon$, the new root $x_3$ will be much larger than the two starting roots $|x_3| \gg |x_{1,2}|$. Visualize how the "third root" depends on the (complex) parameter $\varepsilon$.

**d)** Implement a one-liner that finds all of the roots of a polynomial $p(x)$ of degree $n$ via the following iteration (see [1095✱], [211✱], [981✱], [1650✱], [1346✱], [1355✱], [1743✱], [1450✱], [1950✱], [737✱], [947✱], [1858✱], [784✱], [866✱], [1444✱], [1446✱], [1955✱], [84✱], [1892✱], [786✱], [1789✱], [860✱], [1648✱], [1445✱], [1447✱], [1007✱], [1781✱], and [210✱], and for higher-order convergent methods using the same principle, see [875✱], [1414✱], [1446✱], [1451✱], and [83✱]).

$$x_j^{(i+1)} = x_j^{(i)} - \frac{p(x_j^{(i)})}{\prod_{\substack{k=1 \\ k \neq j}}^{n} (x_j^{(i)} - x_k^{(i)})}$$

$j = 1, \ldots, n$ where $x_j^{(i)}$ is the $i$th iterate of the $j$th root. The starting roots $x_j^{(0)}$ might be arbitrary complex numbers (for polynomials with only real coefficients having complex roots, some of the starting values must be complex numbers).

In Section 3.7 of the Programming volume [1806✱], some graphics showing fractals arising from iterating the Newton method were shown [1017✱], [314✱]. Can the Weierstrass iteration also produce fractals?

**e)** Take a polynomial and calculate its roots. Then, take these roots (in all possible combinations) as coefficients for new polynomials. Calculate their roots, and take them as coefficients for new polynomials, and so on. Iterate this process a few times and show graphically all of the roots in the complex plane.

**f)** Use a "random" parametrized polynomial $p_\tau(z)$ to make an animation of the convergence of the Newton iterations $z \to z - p_\tau(z) / p'_\tau(z)$ (the prime denotes differentiation with respect to $z$) as a function of the starting value $z_0$.

**g)** The Taylor series expansion of a function $f(x)$ around $x_0$ using the Lagrange form of the remainder is

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!} + \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\tilde{x}).$$

where $x_0 \leq \tilde{x} \leq x$.

Choose random polynomials of degree $d$ for $f(x)$. Fix $x$ and make histograms that show the distribution of $\tilde{x}$ as a function of $n$.

### 16.[L2] Nodal Line of sin(24 *x*) sin(*y*) + 6 / 5 sin(*x*) sin(24 *y*) = 0

**a)** The curve defined implicitly by $\sin(24\,x)\sin(y) + 6/5\sin(x)\sin(24\,y) = 0$ is free of self-intersections inside the region $0 < x < \pi$, $0 < y < \pi$. Use this fact to construct a very high-resolution contour plot analogous to

```
ContourPlot[Sin[24 x] Sin[y] + 6/5 Sin[x] Sin[24 y],
            {x, 0, Pi}, {y, 0, Pi}, Contours -> {0},
            PlotPoints -> {bigIntegerSay1000}];
```

without using `ContourPlot` (and without reprogramming `ContourPlot`) by solving differential equations for the nodal curve(s). (The function to be visualized is an eigenfunction of the Helmholtz operator on the square; in comparison to most examples treated in Solution 3 of Chapter 3 of the Graphics volume [1807✱], this times the weights of the two eigenfunctions (with the same eigenvalue) are slightly different, and, as a result, self-intersections of the nodal curve become atypical in this case; see [443✱], [1258✱], and [774✱].)

**a)** Consider the nonlinear Bloch equations [1847✱]

$$S_x'(t) = \kappa\, S_y(t)\, S_z(t)$$
$$S_y'(t) = S_z(t) + \kappa\, S_x(t)\, S_z(t)$$
$$S_z'(t) = -S_y(t).$$

Show how the solution curves $\{S_x(t), S_y(t), S_z(t)\}$ depend on $\kappa$. Choose random initial conditions on the unit sphere $S_x(0)^2 + S_y(0)^2 + S_z(0)^2 = 1$.

## 17.[L1] Branch Cuts of an Elliptic Curve, Strange 4D Attractors

**a)** Make a picture of the branch cuts of the function

$$\sqrt{z \prod_{j=0}^{6} \left(z - \frac{1}{4} e^{\frac{2 i \pi j}{7}}\right) \prod_{j=0}^{4} \left(z - \frac{1}{2} e^{\frac{2 i \pi j}{5} + \frac{i \pi}{10}}\right) \prod_{j=0}^{2} \left(z - \frac{3}{4} e^{\frac{2 i \pi j}{3} + \frac{i \pi}{6}}\right)}.$$

First, solve it by using `ContourPlot`. Second, solve a differential equation for the location of the branch cut and then display the result of solving the differential equation.

**b)** Find systems of coupled nonlinear ODEs of first order $x_i'(t) = p_i(x_1(t), x_2(t), x_3(t), x_4(t))$, $i = 1, 2, 3, 4$ whose solutions exhibit strange attractors in 4D. Use low-order polynomials for the $p_i$.

## 18.[L1] Differently Colored Spikes, Billiard with Gravity

**a)** Color the various spikes in the following picture differently.

```
f[φ_, ϑ_] := Function[r, (Sign[#] Abs[#]^(5/3))& /@
           N[r {Cos[φ] Sin[ϑ], Sin[φ] Sin[ϑ], Cos[ϑ]}]][N @
             Abs[(3465/16(1 - Cos[ϑ]^2)^(3/2)*(221Cos[ϑ]^6 -
                  195Cos[ϑ]^4 + 39Cos[ϑ]^2 - 1)) Cos[4φ]]]

ParametricPlot3D[f[φ, ϑ], {φ, 0, 2Pi}, {ϑ, 0, Pi},
                 PlotPoints -> {41, 41}, Compiled -> False,
                 PlotRange -> All, BoxRatios -> {1, 1, 1},
                 Boxed -> False, Axes -> False];
```

**b)** Consider a point particle in 2D under the influence of gravity (acting downward) being repeatedly and ideally reflected from the curve

$$y(x) = -\sum_{k=-\infty}^{\infty} \theta\left(x - k + \frac{1}{2}\right) \theta\left(k + \frac{1}{2} - x\right) (-1)^k \sqrt{\frac{1}{4} - (x-k)^2}.$$

Visualize some qualitatively different trajectories for various initial conditions.

## 19.[L2] Schwarz–Riemann Minimal Surface, Jorge–Meeks Trinoid, Random Minimal Surfaces

**a)** Make a picture of the following Schwarz–Riemann minimal surface [1361✱], [1638✱], [623✱], [624✱], and [1312✱] (see also Subsection 1.5.2 of the Symbolics volume [1808✱]):

$$\{x(s, t), y(s, t), z(s, t)\} = \mathrm{Re}\left(\int_0^{s+i t} \frac{1}{\sqrt{1 - 14\,\omega^4 + \omega^8}} \{1 - \omega^2, i(1+\omega^2), 2\,\omega^2\}\, d\omega\right).$$

The parameters $s$ and $t$ are from the region of the $s,t$-plane where the following four circles overlap: $(s \pm 2^{-1/2})^2 + (t \pm 2^{-1/2})^2 = 2$. Carry out all calculations numerically. The resulting surface can be smoothly continued by reflecting the surface across the lines that form its boundary. Carry out this continuation.

**b)** Make a picture of the following minimal surface (Jorge–Meeks trinoid) [133✱], [1567✱], [1179✱] and [1379✱] (see also Subsection 1.5.2 of the Symbolics volume [1808✱]):

$$\{x(s,t),\ y(s,t),\ z(s,t)\} = \mathrm{Re}\left(\int_0^{s+i\,t} \frac{1}{(\omega^3-1)^2}\,\{1-\omega^4,\ i(1+\omega^4),\ \omega^2\}\,d\omega\right).$$

$\omega = s + i\,t,\ s,\ t \in \mathbb{R}$.

Carry out all calculations numerically. The Jorge-Meeks trinoid has a threefold rotational symmetry and four-mirror symmetry plane. Make use of this symmetry in the construction. A region in the $s,t$-plane that generates one part of the surface is in polar coordinates given by $0 \le r < 1,\ 0 \le \varphi \le \pi/3$.

**c)** For the following pairs of functions $f(\xi)$ and $g(\xi)$, calculate numerically the parts of the surfaces (for some suitable $r_0$) that are defined via

$$\{x(r,\varphi),\ y(r,\varphi),\ z(r,\varphi)\} = \mathrm{Re}\left(\int_{r_0}^{r\exp(i\varphi)} \{(1-g(\xi)^2),\ i(1+g(\xi)^2)\,f(\xi),\ 2\,f(\xi)\,g(\xi)\}\,d\xi\right).$$

If the functions $f(\xi)$ and $g(\xi)$ have branch cuts, continue the functions to the next Riemann sheet to avoid discontinuities.

This is the list of function pairs:

$$f(\xi) = \xi - \frac{67}{78} \qquad g(\xi) = \left(\frac{1}{\xi}-1\right)^2 + \xi^2 + 2\,\xi + \frac{44}{31}$$

$$f(\xi) = \xi + \frac{14}{27} + \frac{1}{\xi^2} \qquad g(\xi) = \frac{1}{\xi^{5/3}}$$

$$f(\xi) = \xi^3 + 2\,\xi + \left(\frac{1}{\xi^{4/3}}-1\right)^2 + \frac{2}{3} \qquad g(\xi) = 2\,\xi + \frac{112}{69}$$

$$f(\xi) = -2\,\xi^4 \qquad g(\xi) = \frac{\left(\frac{1}{\xi^{5/3}}-1\right)^2(\xi-2)}{\xi^3} - 2$$

$$f(\xi) = -\frac{1173}{310}\,\xi^{-3} \qquad g(\xi) = \xi^3$$

$$f(\xi) = \left(2 + \frac{1}{\xi^6}\right)^2 \qquad g(\xi) = -\frac{28}{55}\,\xi\,(\xi^{5/3}+1)^2$$

$$f(\xi) = \sqrt[3]{\xi + \frac{28}{9}} + 2 \qquad g(\xi) = \sinh\left(\frac{1}{\xi^{4/3}}\right)$$

$$f(\xi) = \frac{\xi - \frac{7}{12}}{\xi^3} \qquad g(\xi) = \frac{61}{56}\,\xi^{7/3}$$

$$f(\xi) = -\frac{\ln(2)}{\ln\left(\frac{1}{\xi^2+1}\right)} \qquad g(\xi) = -\frac{20}{17\,\xi^2}$$

$$f(\xi) = \frac{i\,\pi}{\ln\left((\xi^{5/3}+1)^{-2}\right)} \qquad g(\xi) = \left(2\,\xi + \frac{4}{5}\right)^{\frac{1}{\xi^2}-1}$$

$$f(\xi) = \left(\frac{\ln^{\frac{32}{9}}(\xi)}{3000000} + 1\right)^{-2} \qquad g(\xi) = \frac{\xi^3}{\left(\frac{1}{4\,\xi^{4/3}}-1\right)^2}.$$

### 20.[L2] Precision Modeling, `GoldenRatio` Code from the Tour, Resistor Network

**a)** Model the following curve (as a function[al] of `f[x]`):

```
f[x_] := 2 - x - 5x^2 + 4x^3 + 3x^4 - 2x^5;

SetPrecision[Round, False];

ListPlot[Table[{x, Precision[f[SetPrecision[x, 25]]]}, {x, -5, 5, 10/1001}],
         PlotRange -> {{-5, 5}, {20, 30}},
         Frame -> True, PlotJoined -> True, Axes -> False];
```

**b)** Why do the following two definitions below from page 18 of *The Mathematica Book* [1933✱] really work and give as a result the value of `GoldenRatio` to $k$ digits? Should not there be a small loss of precision in every step of the `FixedPoint` calculation?

```
g1[k_] := FixedPoint[N[Sqrt[1 + #], k]&, 1]
g2[k_] := 1 + FixedPoint[N[1/(1 + #), k]&, 1]
```

**c)** Predict the first few digits of the number calculated by the following sequence of inputs.

```
data = Table[If[Not[IntegerQ[n/10]],
 {n, Coefficient[Fit[Table[Plus @@ IntegerDigits[n^k], {k,  100}],
                     {1, x}, x], x]}, Sequence @@ {}], {n, 1000}];

fit = Fit[data, {Log[10, n]}, n]
```

**d)** The resistance $R_{m,n}$ between the lattice point {0, 0} and the lattice point $\{m,\ n\}$ of an infinite square lattice with unit resistors between lattice points obeys the following set of equations for nonnegative $n$, $m$ [449✱], [450✱], [451✱], [1853✱]:

$$R_{m+2,m+2} = 4\,\frac{m+1}{2\,m+3}\,R_{m+1,m+1} - \frac{2\,m+1}{2\,m+3}\,R_{m,m}$$
$$R_{n+2,n+1} = 2\,R_{n+1,n+1} - R_{n+1,n}$$
$$R_{m+2,0} = -R_{m,0} + 4\,R_{m+1,0} - 2\,R_{m+1,1}$$
$$R_{m+1,n} = -R_{m,n} - R_{m+1,n-1} - R_{m+1,n+1} + 4\,R_{m+1,n} \text{ if } 2 < n < m+1$$
$$R_{n,m} = R_{m,n}$$

The initial conditions for the recursion are $R_{0,0} = 0$, $R_{1,0} = 1/2$, and $R_{1,1} = 2/\pi$.

Visualize $R_{n,m}$ for $0 \le n,\ m \le 200$. In which direction is the resistance (for a fixed distance) the largest? For large distances, the following asymptotic expansion holds:

$$R_{n,m} \xrightarrow[\sqrt{n^2+m^2}\to\infty]{} \frac{1}{\pi}\left(\log\left(\sqrt{m^2+n^2}\right) + \gamma + \frac{\log(8)}{2}\right).$$

In which direction does this expansion hold best?

### 21.$^{L3}$ Auto-Compiling Functions, Card Game

**a)** Given some function definitions for a symbol $f$ (such as `f[x_, y_] :=...`, `f[x_, y_, z_] :=...`), implement a function to be called on $f$ such that subsequent calls to $f$ with specific numeric arguments generate and use compiled versions of the appropriate definitions. Calls to $f$ with uncompilable arguments should use the original definitions for $f$.

**b)** Consider the following card game [176✱]: Two players each get $n$ cards with unique values between 1 and $2\,n$. In each round, each player selects one card randomly from their pile. The player with the smaller card value wins both cards. The game ends when one player runs out of cards.

If possible, speed up the following implementation of the modeled game (*A* and *B* are the two initial lists of card values).

```
cardGameSteps1 = Compile[{{A, _Integer, 1}, {B, _Integer, 1}},
Module[{a = A, b = B, ra, rb, (* round counter *) ℂ = 0},
       While[a != {} && b != {}, ℂ++;
             (* select two random cards *)
             ra = Random[Integer, {1, Length[a]}];
             rb = Random[Integer, {1, Length[b]}];
             (* compare cards and add new card to one player;
                remove second card from other player *)
             If[a[[ra]] > b[[rb]],
                b = Append[b, a[[ra]]]; a[[ra]] = a[[-1]];
                                        a = Drop[a, -1],
                a = Append[a, b[[rb]]]; b[[rb]] = b[[-1]];
                                        b = Drop[b, -1]]];
        (* return number of rounds *) ℂ]];
```

For $n = 10$ carry out $10^6$ games and calculate the average length of the game.

### 22.$^{L2}$ Path of Steepest Descent, Arclength of Fourier Sum, Minimum-Energy Charge Configuration

**a)** For the spiral minimum search problem from Section 1.9, find the minimum by following the path of steepest descent until one reaches the minimum.

**b)** Consider the partial sums $\sum_{k=1}^{n} \sin(k\,x)/k$ of the Fourier series of the function $f(x) = \pi/2 - x/2$. As $n \to \infty$, the arclength of the graph of the Fourier series diverges [1735✱], [1494✱]. How many terms does one have to take into account so that the arclength of the graph of the Fourier series is equal to or greater than twice the arclength of $f(x)$?

**c)** Consider $48\,n$ ($n = 1, 2, \ldots$) point charges on a sphere. Enforce the symmetry group of the cube (that has 48 elements) on the charges and find minimum energy configurations for the charge positions for small $n$. Compare results and timings for various method option settings. Calculate the minimum energy configuration for $n = 36$.

### 23.$^{L2}$ `N[`*expr*`,` *prec*`]` Questions and Compile Questions

**a)** Might it happen that `N[`*expr*`,` *prec*`]` ($prec < \infty$) returns a result with infinite precision for a `NumericQ` *expr*?

**b)** Predict the result of the following input.

```
Precision[SetAccuracy[10^+30 Pi, 50]] -
Accuracy[SetPrecision[10^+30 Pi, 50]] +
Precision[SetAccuracy[10^-30 Pi, 50]] -
Accuracy[SetPrecision[10^-30 Pi, 50]]
```

**c)** Predict the result of the following input.

```
Log[10, Abs[N[SetPrecision[SetPrecision[Pi, 50], Infinity]/Pi - 1, 30]]] < -50
```

**d)** Construct two functions built from elementary functions (like `Log`, `Exp`, `Sqrt`, `Power`, `Sin`, `Cos`, …) $f_1(x)$ and $f_2(x)$, such that the precision $f_1(x)$ is more than ten times the precision of the argument $x$, such that the precision of $f_2(x)$ is less than one-tenth of the precision of the argument $x$.

**e)** For a numerical expression *expr*, `N[`*expr*`, `*prec*`]` (with *prec* > `$MachinePrecision`) typically gives a result that is correct to precision *prec*. Try to construct an expression *expr* such that:
- returns true for `NumericQ[`*expr*`]`
- is built from elementary functions (like `Log`, `Exp`, `Sqrt`, `Power`, `Sin`, `Cos`, …)
- is not identically zero
- does not give any `N::meprec` messages when `N[`*expr*`, 50]` is evaluated
- gives a result for `N[`*expr*`, 50]` that is wrong in the first digit already.

(Do not use `Unprotect`, or set unusual `UpValues`, ….)

**f)** Typically, doing a calculation with machine numbers is faster than doing a calculation with high-precision numbers. Find a counter example to this statement.

**g)** Find a symbolic numeric expression *expr* (meaning `NumericQ[`*expr*`]` yields `True` and `Precision[`*expr*`]` gives `Infinity`), that contains only analytic functions and that, when evaluated, gives `N::meprec` or `Divide::infy` messages.

**h)** Why do the following two inputs give different results?

```
Compile[{x}, 2/Exp[x]][1000]
```

```
Compile[{x}, Evaluate[2/Exp[x]]][1000]
```

**i)** Explain the look of the following three plots.

```
Plot[FractionalPart[Exp[n]],{n, 0, 200},
     Frame -> True, Axes -> False, PlotRange -> All];
```

```
f[n_?NumberQ] := FractionalPart[Exp[SetPrecision[n, Infinity]]];
```

```
Plot[f[n], {n, 0, 200}, Frame -> True, Axes -> False, PlotRange -> All];
```

```
$MaxExtraPrecision = 1000;
Plot[f[n], {n, 0, 2000}, Frame -> True, Axes -> False, PlotRange -> All];
```

**j)** For most inputs *input*, the compiled version `Compile[{}, `*input*`][]` will give the same result as the uncompiled one. Find an example where the compiled version gives a different result.

**k)** Predict the result of the following input.

```
Round[E - w[1] /. NDSolve[{w'[z] == (x /. FindRoot[
          (y /. FindMinimum[-Cos[y - x], {y, x + Pi/8}][[2]]) == w[z],
          {x, 0, 1}]), w[0] == 1}, w, {z, 0, 1}][[1]]]
```

Avoid the premature evaluation of the arguments of the numerical functions.

**l)** Find three real numbers *a*, *b*, and *c* such that three expressions *a* `===` *b*, *a* `===` *c*, and *b* `===` *c* all give `True`, but `Union[{`*a*`, `*b*`, `*c*`}]` returns `{`*a*`, `*b*`, `*c*`}`.

**m)** Predict the result of the following input.

```
Precision[Im[SetPrecision[N[#, 200]& /@
           Unevaluated[10^100 + 10^-10 I], 200]]]
```

**n)** Find an algebraic expression (containing arithmetic operations and one-digit integers) $\xi$ that is zero such that `N[`$\xi$`]` gives a result value whose magnitude is larger than 1.

**o)** Predict the result of the following input.

```
Compile[{{Pi, _Real}}, Pi][2]
```

**p)** Will evaluating the following input give `True`?

```
 N[FindRoot[1]] - (FindRoot[N[1]]) === 0
```

**q)** Given the following definition for the function `f`, find an argument $x$, such that evaluating `f[`$x$`]` emits a `N::meprec` message when evaluating the right-hand side of the function definition.

```
f[x_Real] := N[x, $MachinePrecision]
```

**r)** Find an (analytic in the function-theoretic sense) integrand $int(x)$, such that `NIntegrate[`$int(x)$`, {x, 0, Infinity}]` does not issue any messages and returns a result that is twice the correct value.

**s)** Devise exact rational numbers $x_k$, such that the expression

`N[1, 20] + N[`$x_1$`, 20] + ⋯ + N[`$x_n$`, 20] == N[1 + `$x_1$` + … + `$x_n$`, 20]`

would evaluate to `False`. (Assume that `$MachinePrecision` is less than 20.)

**t)** Guess the shape of the graphic produced by the following input. What exactly does the input do?

```
squareRootOf3CF = With[{p = $MachinePrecision - 2},
With[{𝕣 = (# (1 + Random[Real, {-1., 1.} 10.^-p]))&},
     Compile[x, FixedPoint[
      Function[ξ, 𝕣[𝕣[𝕣[ξ]/𝕣[2.]] + 𝕣[𝕣[3.]/𝕣[2.]/𝕣[ξ]]]], x]]]];

roots = Table[squareRootOf3CF[1.], {10^5}]

Show[Graphics[
Polygon[{{#1 - 1/2, 0}, {#1 + 1/2, 0}, {#1 + 1/2, #2},
          {#1 - 1/2, #2}}]&[First[#], Length[#]]& /@
             Split[Round[(Sqrt[3] - Sort[roots])*
                         10^($MachinePrecision - 3/2)]]],
     Frame -> True];
```

**u)** Implement an optimized version of the following function `f`. Visualize `f[1/GoldenRatio, 10^5, 10^5]`, `f[1/Pi, 10^5, 10^5]`, and `f[1/E, 10^5, 10^5]`.

```
f[x_?(0 < # < 1&), p_Integer, n_Integer] :=
MapIndexed[#1^(1/#2[[1]])&, Rest[
FoldList[Times, 1, Rest[First /@ Rest[
NestList[{FromDigits[{Last[#], 0}, 2.],
           Drop[Last[#], Position[Last[#], 1, {1}, 1][[1, 1]]]}&,
          {1, RealDigits[x, 2, p][[1]]}, n]]]]]];
```

**v)** Find two approximative numbers $z_1$ and $z_2$ and a numerical function $f$, such that $z_1$`===`$z_2$ returns `True`, but $f(z_1)$`===`$f(z_2)$ returns `False`. Can one find examples for high-precision numbers $z_1$ and $z_2$ and a function $f$ continuous in the neighborhood of $z_1$, $z_2$?

**w)** Find a short (shorter than 20 characters) input that issues a `N::meprec` message.

**x)** As mentioned in the main text, linear algebra functions operating on high-precision matrices use internally fixed-precision to carry out the calculations to avoid excessive cancellations. As a result of using fixed-precision arithmetic, the resulting digits of all numbers are no longer guaranteed to be correct. Find an example of a $3\times 3$ high-precision matrix, where `Inverse` applied to this matrix results in matrix elements with incorrect digits.

**y)** Predict the result of the following input.

```
(ArcTan[10^100] - Pi/2)^0``100
```

**z)** Predict the result of the following input:

```
f[x_] := x/Sqrt[x^2] Exp[-1/Sqrt[x^2]]
f[x /; x == 0] = 0

FindRoot[f[x/10] == 0, {x, 1}]
```

How does one set the options of `FindRoot` to get the zero $x_0 = 0$ of `f[x/10]` within $|x_0| < 10^{-6}$?

**$\alpha$)** Why does the following input give a message?

```
Module[{c = 0.5436268955915372089486, g},
       g[x_] := x + c;
       NIntegrate[g[x] - 1/g[x], {x, 0, 1}]]
```

### 24.[L2] Series Expansion for Anharmonic Oscillator

**a)** After making the substitution $y(x) = \exp(-x^3/3)\, u(x)$ [695✱], [58✱], [779✱], [708✱] for $y(x)$ in

$$-y''(x) + x^4\, y(x) = \lambda\, y(x)$$

we get a new differential equation for $u(x)$. Making for $u(x)$ a power series ansatz of the form $u(x) = \sum_{i=0}^{\infty} a_i\, x^i$ calculate the first 100 $a_i$. Determine $\lambda$ such that the first zero in $x$ of $\exp(-x^3/3) \sum_{i=0}^{100} a_i\, x^i$ is a double zero.

**b)** After making the substitution $y(x) = \exp(-x^2/2)\, u(x)$ for $y(x)$ in

$$-y''(x) + (x^2 + x^4)\, y(x) = \lambda\, y(x)$$

we get a new differential equation for $u(x)$ [1822✱], [1823✱]. Making for $u(x)$ a power series ansatz of the form $u(x) = \sum_{i=0}^{\infty} a_i(\lambda)\, x^i$ calculate the first 200 $a_i$. Determine $\lambda$ such that the first zero in $\lambda$ of $a_i(\lambda)$. How many correct digits does one get from $a_{200}$?

**c)** Generate $n$ terms of the power series solution $y_n(x) = \sum_{i=0}^{n} a_i(\lambda)\, x^i$ of the differential equation $-y''(x) + x^4\, y(x) = \lambda\, y(x)$, $y(\pm\infty) = 0$. Determine the upper and lower bounds for the lowest possible $\lambda$ by finding high-precision approximations for the zeros of $y_n(x^*)$ and $y_n'(x^*)$ for "suitably chosen" $x^*$. Find an approximation for the lowest possible $\lambda$ that is correct to 1000 digits [1809✱].

### 25.[L1] Gibbs Distributions, Optimal Bin Size, Rounded Sums, Odlyzko-Stanley Sequences

**a)** Model the approach to equilibrium of an ensemble of pairwise interacting particles. Let a set of $n$ particles, each initially having energy $e_0$, be given. For all particles, allow only (positive) integer-valued energies. Model a two-body collision by taking two randomly selected particles and randomly redistribute their (common) energy between these two particles [599✱], [523✱], [370✱].

**b)** In [602✱], [603✱], the following statistical mechanics-inspired method for the generation of $n$ normally distributed random numbers was given: Prepare a list of $n$ approximative 1's. Randomly (with uniform probability distribution) select two nonidentical integers $i$ and $j$. Update the $i$th and $j$th elements $l_i$ and $l_j$ according to $l_i' = \mathfrak{r}(l_i + l_j)\big/\sqrt{2}$,

$l'_j = (2\,\mathbb{r}^2\, l_j - l_j - l_j)\big/\sqrt{2}$. Here $\mathbb{r}$ is a random variable with values $\pm 1$. Repeat this updating process about $3/2\log(n)\,n$ times. Then, one obtains $n$ normally distributed random numbers with probability distribution $p(x) = (2\,\pi)^{-1/2}\exp(-x^2/2)$.

Implement a compiled version of this method for generating normally distributed random numbers. Compare the resulting distribution with the ideal one. How long does it take to generate $10^6$ numbers in this way? Compare with the direct method that uses the inverse error function `InverseErf`.

**c)** Generate $10^4$ sums of $10^3$ uniformly distributed numbers. Use the central limit theorem to approximate the distribution function of the sums. Which bin size results in a minimal mean square difference between an approximative histogram density and the limit distribution?

**d)** Typically, the sum of rounded summands does not coincide with the rounded sum of summands. Let $\xi_k$ be uniformly distributed random variables from $[-L, L]$ and $\zeta_k$ their rounded nearest integers in base $b$ (assume $L \geq 1/b$). Then the probability $p_n^{(b)}$ that the two sums $\sum_{k=1}^n \xi_k$ and $\sum_{k=1}^n \zeta_k$ rounded to the nearest multiple of $b$ coincide is [1183✱]

$$p_n^{(b)} = \frac{2}{\pi\, b}\int_0^\infty \left(\frac{\sin(x)}{x}\right)^{n-1} \frac{\sin^2(b\,x)}{x^2}\,(1 + \delta_{n \bmod 2,0}\cos(x))\,dx.$$

Form a random sum with $n = 10^6$ terms that confirms this probability for $b = 10$ within $10^{-5}$.

**e)** Implement a function that returns all elements of the Odlyzko-Stanley sequence $\mathcal{S}_k$ that are less than a given integer $n$. The Odlyzko-Stanley sequence $\mathcal{S}_k$ is the sequence of integers $\{a_0, a_1, \ldots, a_j, \ldots\}$ with $a_0 = 0$, $a_1 = k$, and $a_j$ defined implicitly through the condition that it is the smallest integer, such that the sequence $\{a_0, a_1, \ldots, a_{j-1}\}$ does not contain any subsequence of three elements that form an arithmetic progression [684✱]. Visualize the resulting sequences and local averages of the sequences for $k = 1, \ldots, 13$ and $n = 10^6$.

### 26.[L1] Nesting Tan, Thompson's Lamp, Digit Jumping

**a)** Explain the "steps" visible in the following picture.

```
nl = NestList[Tan, N[2, 200], 2000];
precList = Precision /@ nl;
ListPlot[precList];
```

**b)** Explain some characteristics of the following graphic.

```
fl = Rest[FoldList[Times, 1., Table[Tan[k], {k, 2 10^5}] // N]];
ListPlot[Log @ Abs[fl], PlotRange -> All, PlotStyle -> {PointSize[0.002]}]
```

**c)** Predict whether in the following attempt to model Thompson's lamp [1774✱], [168✱], [191✱], [781✱], [750✱], [536✱], [463✱], [1749✱], [1467✱], [1656✱], [436✱] the lamp be on or off "at the end"? Do not run the code to find out.

```
t = 0; δt = 1/2; lampOnQ = True;
While[t != 1``10000, t = t + δt; δt = δt/2; lampOnQ = Not[lampOnQ]]
lampOnQ
```

What happens when one replaces `` 1``10000 `` by the exact integer `1`?

**d)** What does the following code do and what is the expected shape of the resulting plots?

```
cf = Compile[{{n, _Integer}},
Table[Module[{digits, λ = 100, k = 1, sOld = 1, sNew = 1},
           digits = Join[{Random[Integer, {1, 9}]},
                         Table[Random[Integer, {0, 9}], {λ}]];
           While[k++; If[sOld > λ,
```

```
                            digits = Join[digits,
                                Table[Random[Integer, {0, 9}], {sOld}]];
                    λ = Length[digits]];
                    sNew = sOld + digits[[sOld]];
                    sOld =!= sNew, sOld = sNew];
     {k, sNew}], {n}]];

data = cf[10^6];

Show[GraphicsArray[
ListPlot[{First[#], Log @ Length[#]}& /@ Split[Sort[# /@ data]],
         PlotRange -> All, DisplayFunction -> Identity]& /@ {First, Last}]];
```

### 27.[L2] Parking Cars, Causal Network, Seceder Model, Run Lengths, Cycles in Random Permutations, Iterated Inner Points, Exchange Shuffling, Frog Model, Second Arcsine Law, Average Brownian Excursion Shape

**a)** Make a Monte-Carlo simulation of the following problem: Cars of length 1 are parked randomly inside a linear parking lot of length $l$ (consider the cases $l = 100$, $l = 1000$). What is the expected number of cars that fit into the lot? Compare with the theoretical result [1012✱], [1436✱], [1478✱] for large $l$

$$expectedNumberOfCars = c\,l - (1-c) + O\left(\frac{1}{l}\right)$$

where $c$ is given by

$$c = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,du\right)dt.$$

Write a version of the simulation that makes use of `Compile`.

**b)** Find a fast method to calculate the $\psi_n^m$ $(-1000 \le m \le m,\ 0 \le n \le 1000)$ that obey the following equations [905✱]:

$$\psi_n^m\,\psi_n^{m+1}\,\psi_n^{m-1}\,\psi_{n+1}^m = 1 \qquad 0 \le n < \infty,\ -\infty < m < \infty$$
$$\psi_0^m = 1 - 2\,\delta_{m,0}$$
$$\psi_1^m = 1.$$

**c)** The seceder model [516✱], [1705✱] (used to model the spontaneous formation of groups) is the following: Starting with a list $l$ of $n$ zeros, one step consists of $n$ iterations of an update step. In the update step, randomly three elements of $l$ are selected. Then the element of the three that has the largest distance from the average is chosen. A random real number (say uniformly distributed in $[-1, 1]$) is added the selected element and the resulting value replaces a randomly chosen element of $l$. All other elements of $l$ stay unchanged.

Implement the seceder model efficiently. Run and visualize 1000 update steps of a starting list of length 1000. Is a run and a visualization of 10000 update steps of a starting list of length 10000 doable?

**d)** A run in a list of numbers $\{n_1, n_2, \ldots, n_k\}$ is a sublist of consecutive increasing numbers $\{n_i, n_{i+1}, \ldots, n_{i+l}\}$. For 1000 random permutations, each of length 10000, calculate how many runs of length $\lambda$ occurred. Use `Compile` for the generation and analysis of the random permutation.

**e)** Calculate 1000 random permutations of length 1000, and analyze the number of cycles of these permutations. Use `Compile` for the generation and analysis of the random permutation.

**f)** Given a permutation $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of the integers $\{1, 2, \ldots, n\}$, and a permutation $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$ of the integers $\{1, 2, \ldots, k\}$ (where $k \le n$), one says the pattern $\tau$ occurs in the permutation $\sigma$ if there exists a sequence of indices $\{j_1, j_2, \ldots, j_k\}$, such that $\sigma_{j_m} \lessgtr \sigma_{j_n}$ whenever $\tau_{j_m} \lessgtr \tau_{j_n}$ for all $1 \le m < n \le k$. For each of the $4!$ different patterns of $\{1, 2, 3, 4\}$, find how many of the $8!$ permutations of $\{1, 2, \ldots, 8\}$ do not include the search pattern? Can you calculate the same for the $10!$ permutations of $\{1, 2, \ldots, 10\}$? Aim for a memory- and time-efficient implementation.

**g)** The cut sequence [1101✱], [1102✱] of a permutation of the first $n$ integers $\{k_1, k_2, \ldots, k_n\}$ is a list of lists of the form $\{\{k_1, k_2, \ldots, k_{i_1}\}, \{k_{i_1+1}, \ldots, k_{i_2}\}, \{k_{i_{2+1}}, \ldots, k_{i_3}\}, \ldots, \{k_{i_{m+1}}, \ldots, k_n\}\}$ such that $k_{i_l} < k_j$ for all $i_l < j$. (So, for example the cut sequence of $\{1, 2, 6, 7, 4, 5, 3\}$ is $\{\{1\}, \{2\}, \{6, 7, 4, 5, 3\}\}$.) Given a permutation, implement an efficient calculation of its cut sequence. For 10000 random permutations of 1000 integers, calculate the average length of the cut sequence.

**h)** Generate an $n \times n$ matrix **A** (say, $n = 256$) with $p$% 0's and $(1 - p)$% 1's. Now iterate the following process until all 0's have been transformed to 1's: For each element $a_{ij}$ of the matrix that is 0, check if its two left and right and its two upper and lower neighbors are 0. If they are not, change the element to 1; if they are, do nothing. Visualize how the 1's form in the iterations. Use $p = 90$, $p = 99$, and $p = 99.9$.

**i)** Model the following (exchange) shuffle process: Given a list of integers $\{1, 2, \ldots, n\}$, exchange the first integer with a randomly selected one from the list (maybe with itself). Then exchange the second one with a randomly selected one, ..., then exchange the last one with a randomly selected one [1550✱], [1620✱]. Model $10^6$ such exchange shuffles and analyze the probabilities of the resulting permutations for $n = 4, 5, 6$. Compare with the exact probabilities.

**j)** Consider the so-called frog model [45✱], [1479✱] of statistical mechanics. Assume sleeping frogs are placed on the lattice points of a $n \times n$ square lattice. Then one frog awakes and jumps to one of the four neighboring sites. This wakes up the frog on this site and in the next step each of the two frogs jump to a randomly selected neighboring site. Then this process continues. Make an animation that shows how the awakened frogs spread out and carry out enough steps to estimate the distribution of the long-time limit of the probability to find l $0 \le n \le 10$ frogs per site.

**k)** The so-called second arc-sine law says that in a 1D Brownian motion that starts at time 0 at the origin and stops at time $T$, the probability distribution $p(\tau)$ for the largest time $\tau$ the particle visited the origin is $p(\tau / T) = 2 / \pi \arcsin((\tau / T)^{1/2})$ [1965✱], [430✱]. Model a 1D Brownian motion with constant step size and calculate the theoretical probability density with the modeled one.

**l)** An excursion of a 1D Brownian motion $x(t)$ is defined to be a maximal connected part $\{x(t)\}_{T_0 \le t \le T_1}$ of the same sign. The average shape of the (scaled) excursions is $|\overline{x}(\tau)|(T_1 - T_0)^{-1/2} = (8 / \pi\, \tau(1 - \tau))^{1/2}$ where $\tau = (t - T_0)/(T_1 - T_0)$ [122✱], [427✱], [1755✱]. Model a 1D Brownian motion by using constant step size (and ignoring very small excursions) and compare the theoretical probability density with the modeled one.

## 28.[L3] Poincaré Sections, Random Stirring, ABC-System, Vortices on a Sphere, Oscillations of a Triangular Spring Network, Lorenz System

**a)** Make a Poincaré section plot for the following coupled nonlinear system of differential equations:

$$x'(t) = y(t)$$
$$y'(t) = y(t)\, z(t) - x(t)$$
$$z'(t) = 1 - (y(t))^2.$$

(A Poincaré section [933✱] here means a plot of points in the $x$,$y$-plane (this means $z(t) = 0$) formed by the solutions of the differential equations [524✱].) Use two different approaches to obtain the Poincaré sections.

**b)** Predict what the following code will do.

```
rotate = Compile[{{center, _Real, 1}, φ, {points, _Real, 2}},
          Module[{r, p, ϕ, ℛ}, Function[point,
```

```
                p = point - center; r = Sqrt[p.p]; ϕ = Exp[-r] φ;
                ℛ = {{Cos[ϕ], Sin[ϕ]}, {-Sin[ϕ], Cos[ϕ]}};
                ℛ.p + center] /@ points]];

pp = 10000; R = 3/4; n = 3; m = 3;
centers = Table[N[{Cos[j/n 2Pi], Sin[j/n 2Pi]}], {j, 0, n - 1}];
ℒ = Table[R {Cos[φ], Sin[φ]}, {φ, 0, 2Pi, 2Pi/pp}] // N;

SeedRandom[333];
Do[center = centers[[Random[Integer, {1, n}]]];
   φ = Random[Real, Pi {0, 1}];
   Do[𝕃 = rotate[center, k/m φ, ℒ];
      Show[Graphics[Polygon[𝕃],
                    PlotRange -> Max[Abs[𝕃]]{{-1, 1}, {-1, 1}},
                    Frame -> True, FrameTicks -> None,
                    AspectRatio -> Automatic]], {k, 0, m - 1}];
      ℒ = rotate[center, φ, ℒ], {j, 100}];
```

Generalize whatever the code is doing to 3D case.

**c)** Consider the system of coupled differential equations (Arnold–Beltrami–Childress system)

$$x'(t) = \gamma \cos(y(t)) + \alpha \sin(z(t))$$
$$y'(t) = \alpha \cos(x(t)) + \beta \sin(x(t))$$
$$z'(t) = \beta \cos(x(t)) + \gamma \sin(y(t)).$$

Make an animation showing how the family of solutions depends on $0 \le \gamma \le 1$ [714✱], [230✱].

**d)** The equations of motion of $n$ point vortices of strength $\Gamma_k$ on a unit sphere are [1433✱], [994✱], [1115✱], [1588✱]:

$$\mathbf{x}_j'(t) = \sum_{\substack{k=1 \\ k \neq j}}^{n} \Gamma_k \, \frac{\mathbf{x}_k(t) \times \mathbf{x}_j(t)}{1 - \mathbf{x}_k(t).\mathbf{x}_j(t)}.$$

Calculate and visualize some example trajectories of three and four vortices.

**e)** Consider a mechanical system built from point masses and springs. The springs are located along the edges of a recursively subdivided regular triangle and the point masses are the corresponding vertices. For a system consisting of 45 point masses (meaning 108 springs), fix the outermost three point masses and visualize the oscillations of the system for some chosen initial conditions.

Linearize the resulting equations (meaning small oscillations) and visualize some of the eigenoscillations [594✱].

**f)** Consider the classic Lorenz system [1182✱], [1287✱], [89✱], [1752✱], [1099✱]

$$x'(t) = \sigma\,(y(t) - x(t))$$
$$y'(t) = \rho\, x(t) - y(t) - x(t)\, z(t)$$
$$z'(t) = x(t)\, y(t) - \beta\, z(t)$$

with parameters $\sigma = 10$, $\rho = 28$, $\beta = 8/3$ and initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. Sketch how the surface $x(t) = 0$ (as a function of the initial conditions) in $x_0$, $y_0$, $z_0$-space evolves as a function of $t$.

### 29.[L2] Polynomial Coefficient, Fourier Differentiation

**a)** Calculate numerically via contour integration the coefficient of $x^{3000}$ in the expansion of the polynomial $p(x) = (x+1)^{2000}\,(x^2+x+1)^{1000}\,(x^4+x^3+x^2+x+1)^{500}$ to 13 significant digits. Calculate the coefficient in the saddle point approximation [1455✱], [1392✱], [1528✱], [729✱] of the integral. (This problem comes from [593✱].)

**b)** Given the continuous Fourier transform $\hat{f}(k)$ of a function $f(x)$ restricted to the interval $[0, L]$,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(k)]_{[0,L]}\, e^{i k x}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^L f(k)\, e^{i k x}\, dx$$

($[f]_I$ indicates the function $f$ restricted to $I$), how can one approximate the $\nu$th element $\hat{f}_\nu$, $\nu = 1, \ldots, n$ of the discrete Fourier transform

$$\hat{f}_\nu = \frac{1}{\sqrt{n}} \sum_{\mu=1}^{n} f_\mu \exp(2\pi i (\mu-1)(\nu-1)/n)$$

through $\hat{f}(k)$ in the large $n$ limit ($L/n$ sufficiently small)? Here $f_\mu = f(x_\mu)$ and $x_\mu = \mu/n\,L$. Compare the direct result with the asymptotic result for some examples.

For the continuous Fourier transform the differentiation of the original function means multiplication by $(-i\,k)$ for the transformed function. How does this translate to the discrete Fourier transform?

### 30.[L1] Abel Differentials, Implicit Parameter Dependence

**a)** Calculate a numerical approximation for $s(x)$, $0 \le x \le 2$

$$s(x) = \int_0^x \left( \sum_{i=1}^{6} (y_i(x)-1)/(y_i(x)+1) \right) dx.$$

Here, $y_i(x)$ are the six independent roots of the polynomial $y(x)^6 + x^3\, y(x)^2 + y(x) - 4x^4 - 2x - 3 = 0$ considered as smooth functions of $x$.

**b)** Given the ordinary differential equation $w_\zeta'(z) == \cos(w_\zeta(z)) + \sin(\zeta)$, $w_\zeta(0) = \zeta^2$ (prime denotes differentiation with respect to $z$) with parametric dependence on $\zeta$, calculate $\partial^2 w_\zeta(1)/\partial\zeta^2\,|_{\zeta=1}$ to 20 digits.

### 31.[L2] Modular Function Fourier Series, Differential Equation for *J*(*z*), Singular Moduli

**a)** The Dedekind eta function $\eta(z)$ (in *Mathematica* called `DedekindEta`) has a series representation of the form $\eta(z) = \sum_{k=1}^{\infty} a_k\, e^{k i \pi z/12}$ ($a_k = -1$ or 1 or 0). Calculate numerically the first 50 nonvanishing terms of this series. ($\eta(z)$ is defined only in the upper half-plane.)
The Dedekind eta function has the product representation $\eta(z) = e^{\frac{\pi i z}{12}} \prod_{k=1}^{\infty} (1 - e^{2\pi i k z})$. Use this product representation to calculate the first 50 nonvanishing series terms.

**b)** The Klein modular function $J(z)$ has a series representation of the form $J(z) = 1/1728\,(e^{-2 i \pi z} + 744 + \sum_{k=1}^{\infty} a_k\, e^{2 k i \pi z})$ with all $a_k$ being integers. Calculate numerically the first 100 (nonvanishing) terms of this series. ($J(z)$ is defined only in the upper half-plane).
Use two different methods to calculate the terms to make sure that they are correct.
The coefficients $a_k$ of the series can be represented in the following way:

$$a_k = \frac{2\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j} A_j(k)\, I_1\!\left(4\pi \frac{\sqrt{k}}{j}\right)$$

where $I_1(z)$ is the Bessel function `BesselI[1,z]` and

$$A_j(k) = \sum_{\substack{h=0 \\ \gcd(h,j)=1}}^{j-1} \exp(-2\pi i (h k + H(j, h))/j)$$

$$h\, H(j, h) = -1 \bmod j.$$

Calculate $a_{100}$ using this formula.

**c)** The function $w(z)$ (in *Mathematica* `KleinInvariantJ[z]`) fulfills a nonlinear differential equation of the following form ($c_k \in \mathbb{Z}$, $\mathrm{Im}(z) > 0$):

$$c_1\, w'(z)^4 + c_2\, w'(z)^4\, w(z) + c_3\, w'(z)^4\, w(z)^2 + c_4\, w''(z)^2\, w(z)^2 + c_5\, w''(z)^2\, w(z)^3 + c_6\, w''(z)^2\, w(z)^4 + c_7\, w'(z)\, w^{(3)}(z)\, w(z)^2 + c_8\, w'(z)\, w^{(3)}(z)\, w(z)^3 + c_9\, w'(z)\, w^{(3)}(z)\, w(z)^4 = 0.$$

Find the $c_k$.

**d)** For certain values $q^*_{n,j}$ of $q = \exp(2 k i \pi z)$, the Klein modular function $J(q)$ (in *Mathematica,* as a function of $z$, `KleinInvariantJ[z]`) has the property $J(q^*_{n,j}) = J(n\, q^*_{n,j})$ where $n$ is a positive integer (because the values are not unique, we use the additional index $j$). These are the so-called singular moduli and they have the form $q^* = r_1 + i \sqrt{r_2}$ where $r_1$ and $r_2$ are rational numbers, $r_2 \geq 0$ [203✱], [746✱], [500✱], [998✱], [521✱], [423✱]. For $n = 1, \ldots, 20$, find such $q^*_{n,j}$.

## 32.[L2] Curve Thickening, Textures, Seed Growth

**a)** In [14✱], [1177✱], [15✱], the following formula for generating a function $f_\rho[\boldsymbol{c}(t)](z)$ whose absolute value represents a "thickened" version of the 2D curve $c(t) = c_x(t) + i\, c_y(t)$, $0 \leq t \leq T$ (represented in the complex plane) was given:

$$f_\rho[c(t)](z) = \exp\!\left(-\frac{z\,\bar{z}}{\rho^2}\right) \int_0^T |c'(\tau)| \exp\!\left(-\frac{c(t)\,\overline{c(t)}}{\rho^2} + \frac{2\, z\, \overline{c(t)}}{\rho^2} + \frac{1}{\rho^2} \int_0^t \left(c(\tau)\, \overline{c'(\tau)} - c'(\tau)\, \overline{c(\tau)}\right) d\tau\right) dt.$$

The parameter $\rho$ controls the "thickness" of the curve. Visualize the real part, the imaginary part, the absolute value and the argument of the function $f_1[t\, e^{i t}](z)$, $0 \leq t \leq 2\pi$ (a spiral) and $f_1[e^{i t}](z)$, $0 \leq t \leq 2\pi$ (a circle).

**b)** Iteratively carrying out Fourier transforms, list convolution, and other arithmetic operations on lists of numbers can be used to generate textures (for texture mappings, for instance [538✱]). Find examples of such procedures that generate "nice" textures.

**c)** Given a square matrix **A** with nonnegative integer entries $a_{i,j} = k_m(i, j)$, implement a compiled function `growSeeds` that randomly replaces eventually present zeros in **A** with the values of one of the neighboring elements until all zeros are gone. Apply the function `growSeeds` to some matrices of size $100 \times 100$ to $400 \times 400$ that initially contain a few ten to a few hundred randomly distributed nonzero integers, and to matrices whose initial nonzeros form various regular patterns and visualize the results. Highlight the boundaries between matrix elements with different $k_m(i, j)$.

**d)** Given a matrix with 0s and 1s, replace each 0 by the number of neighborhood enlargements it minimally takes until this 0 is reached from an existing 1. For various random and structured initial matrices, visualize the resulting matrices.

### 33.[L2] First Digit Frequencies in Mandelbrot Set Calculation

Implement the numerical calculation of the Mandelbrot set defined as all points $c$ of the complex plane such that the iteration of the map $z \to z^2 - c$ stays bounded. Monitor the first digit of (the real and imaginary parts of) $z$ in all iteration steps for all $z$'s. (This should be done as a "side effect" in the calculation, but it is the main point within this exercise). Calculate how many times the digit $i$ ($i = 1, 2, \ldots, 9$) occurred. Compare the result with Benford's rule [182✱] (see Exercise 1 from Chapter 6 of the Programming volume [1806✱]).

### 34.[L1] Interesting Jerk Functions

A jerk function [1628✱], [1162✱], [1711✱], [1712✱], [1163✱], [268✱], [1209✱], [1713✱], [1714✱], [544✱], [1211✱], [545✱], [1210✱], [726✱], [1939✱], [727✱] is a polynomial $p(t, x(t), x'(t), x''(t))$ used on the right-hand side of the differential equation $x'''(t) = p(t, x(t), x'(t), x''(t))$. (Sometimes jerk functions are defined to be explicitly independent of the independent variable, this means of the form $p(x(t), x'(t), x''(t))$.) Find some jerk functions that generate an "interesting" phase portrait (graph of $\{x(t), x'(t)\}$). Find some jerk functions that have "interesting" extended phase portraits $\{x(t), x'(t), x''(t)\}$.

### 35.[L2] Initial Value Problems for the Schrödinger Equation

In scaled coordinates $\xi$, $\tau$, ($\xi$ being the position, $0 \le \xi \le 1$, $\tau$ being the time), the Schrödinger equation of a free particle in a potential well can be written in the form

$$i \frac{\partial \psi(\xi, \tau)}{\partial \tau} = -\frac{1}{4\pi} \frac{\partial^2 \psi(\xi, \tau)}{\partial \xi^2}.$$

Solve numerically this equation and then use the obtained solution to visualize the result for the four initial-boundary value problems:

1) $\psi(0, \tau) = \psi(1, \tau) = 0$ and $\psi(\xi, 0) = \sum_{k=1}^{5} \sin(k \pi \xi)$ and $0 \le \tau \le 1$

2) $\psi(0, \tau) = \psi(1, \tau) = 0$ and $\psi(\xi, 0) = \sum_{k=1}^{20} \sin(k \pi \xi)$ and $0 \le \tau \le 1$

3) $\psi(0, \tau) = \psi(1, \tau) = 0$ and $\psi(\xi, 0) = \theta(\xi)\, \theta(\frac{1}{2} - \xi)\, e^{5 i \xi} \exp\left(\left((\xi - \frac{1}{4})^2 - \frac{1}{16}\right)^{-1}\right)$ and $0 \le \tau \le 1$

4) $\psi(0, \tau) = \psi(1, \tau) = 0$ and $\psi(\xi, 0) = 1$ and $0 \le \tau \le 1$

5) $\psi(0, \tau) = \psi(1, \tau) = 0$ and $\psi(\xi, 0) = \sum_{k=\overline{k}-\delta k}^{\overline{k}+\delta k} \exp\left(-(k - \overline{k})^2 / (4 \sigma^2)\right) \exp(-i \pi\, k^2\, \tau / 4) \sin(k \pi \xi)$ and
$10^{-3} \le \tau \le 40 \times 10^{-3}$ (chose $\overline{k} \approx 10^3$, $\delta k \approx 10^2$, and $\sigma \approx 10^0$)

For each of the four problems, compare the solution generated by the separation of variable method with the one generated by `NDSolve`.

### 36.[L3] Initial Value Problems for the Wave Equation

Use the function `NDSolve` to numerically solve the following initial value problem for the wave equation in *dim* dimensions (*dim* = 1, 2, 3)

$$\frac{\partial^2 u(t, \boldsymbol{x})}{\partial t^2} = \Delta u(t, \boldsymbol{x}) = \sum_{k=1}^{dim} \frac{\partial^2 u(t, \boldsymbol{x})}{\partial x_k^2}$$

$$\boldsymbol{x} = \{x_1, x_2, \ldots, x_{\text{dim}}\}$$

in the domains $0 \le t \le 3$ and $0 \le |\boldsymbol{x}| \le 3$. Let $u_0(\boldsymbol{x}) = u(0, \boldsymbol{x})$ be concentrated in a ball of radius $R = 1$ around the origin, and let $u_0(\boldsymbol{x}) = \cos^2(\pi |\boldsymbol{x}|/2)$. Let $\frac{\partial}{\partial t} u(t, \boldsymbol{x})\big|_{t=0} = 0$. This choice of the function $u_0(\boldsymbol{x})$ goes smoothly to zero so that singularities will form—contrasted to constant initial conditions inside a ball.

Compare the `NDSolve` solution with the solution obtained by using the following representations of the solutions [1871✱], [1802✱], [625✱], [946✱], [985✱], [1376✱], [140✱], [1149✱], [1477✱]:

One dimension:

$$u(t, x) = \frac{1}{2}(u_0(x+t) + u_0(x-t))$$

Two dimensions:

$$u(t, \boldsymbol{x}) = \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{B_t(\boldsymbol{x})} \frac{u_0(\boldsymbol{\xi})}{\sqrt{t^2 - |\boldsymbol{x} - \boldsymbol{\xi}|^2}} d\boldsymbol{\xi}$$

Here, the integration is carried out over the sphere $B_t(\boldsymbol{x})$ of radius $t$ around $\boldsymbol{x}$.

Three dimensions:

$$u(t, \boldsymbol{x}) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left( \int_{\partial B_1(\boldsymbol{x})} t\, u_0(\boldsymbol{x} + t\,\boldsymbol{\xi})\, d\boldsymbol{\sigma}_\xi \right)$$

Here, the integration is carried out over the surface of a sphere $\partial B_1(\boldsymbol{x})$ of radius 1 around $\boldsymbol{x}$. (For $n$ dimensions, see [576✱].)

### 37.[L1] Kleinian Group, Continued Fractions, Lüroth Expansions, Lehner Fractions, Brjuno Function, Bolyai Expansion

**a)** The two matrices

$$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma^4 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \text{ and } \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

where

$$\gamma = e^{\frac{2\pi i}{7}}, \quad a = \frac{\gamma^5 - \gamma^2}{\sqrt{-7}}, \quad b = \frac{\gamma^3 - \gamma^4}{\sqrt{-7}}, \quad c = \frac{\gamma^6 - \gamma}{\sqrt{-7}}$$

are generators of a finite group with the group composition being matrix multiplication (see, for instance, [470✱]). Calculate numerically the order of the group as well as the complete group multiplication table.

**b)** Define a function `continuedFraction[`*realNumber*`, `*order*`]` that approximates a positive real number *realNumber* (head `Real`) by a continued fraction of depth *order*. Compare with the built-in function `ContinuedFraction`.

For the first 20 integers $n$, determine numerically approximate values for the probability that the integer $n$ appears in the continued fraction of a randomly chosen real number.

Implement a similar function `continuedInverseRoot` that, for a given $m$, generates the $c_k$ in the expansion [1239✱]

$$realNumber = c_0 + \frac{1}{\sqrt[m]{c_1 + \frac{1}{\sqrt[m]{c_2 + \frac{1}{\sqrt[m]{c_3 + \frac{1}{\sqrt[m]{c_4 + \cdots}}}}}}}}.$$

**c)** Let $\{a_1(x),\ a_2(x),\ \ldots,\ a_n(x)\}$ be the first $n$ terms of the continued fraction expansion of the real number $x$. For almost all $x \in (0,\ 1)$ and $n \to \infty$, the probability that $\ln(2)/n \max(\{a_1(x),\ a_2(x),\ \ldots,\ a_n(x)\}) < X$ is given by $\exp(-1/X)$ [656✱]. Carry out a numerical simulation that confirms this result.

**d)** The Lüroth expansion of a real number $x$ $(0 < x < 1)$ is of the form [1440✱], [657✱], [664✱], [1639✱]

$$x = \frac{1}{a_1} + \sum_{k=2}^{\infty} \frac{1}{a_k} \prod_{j=1}^{k-1} \frac{1}{a_j(a_j - 1)} =$$

$$= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)\,a_2} + \frac{1}{a_1(a_1-1)\,a_2(a_2-1)\,a_3} + \cdots + \frac{1}{a_1(a_1-1)\,a_2(a_2-1)\ldots a_{n-1}(a_{n-1}-1)\,a_n} + \ldots.$$

The Lüroth terms $a_n$ can be calculated via:

$$a_n = \alpha(\mathcal{T}^{n-1}(x))$$

$$\alpha(x) = \left\lfloor \frac{1}{x} \right\rfloor + 1$$

$$\mathcal{T}(x) = \left\lfloor \frac{1}{x} \right\rfloor \left( \left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x - \left\lfloor \frac{1}{x} \right\rfloor.$$

How many Lüroth terms are needed to approximate $1/\pi$ to 1000 digits?

**e)** The Lehner continued fraction expansion of a real number $x$ $(1 < x < 2)$ is of the form [1132✱]

$$x = b_0 + \frac{c_1}{b_1 + \frac{c_2}{b_2 + \frac{c_3}{b_3 + \cdots}}}.$$

The integers $\{b_k,\ c_{k+1}\}$ can be calculated via

$$\{b_k,\ c_{k+1}\} = \beta(\mathcal{L}^{k+1}(x))$$

$$\beta(x) = \begin{cases} \{2,\ -1\} & 1 \le x < \frac{3}{2} \\ \{1,\ 1\} & \frac{3}{2} \le x < 2 \end{cases}$$

$$\mathcal{L}(x) = \begin{cases} \frac{1}{2-x} & 1 \le x < \frac{3}{2} \\ \frac{1}{x-1} & \frac{3}{2} \le x < 2. \end{cases}$$

Investigate the structure of $\mathcal{L}^k(x)$ for various real $x$ $(1 < x < 2)$. For high-precision approximations of $\pi/2$ and $e/2$, investigate the probability that $\{b_k,\ c_{k+1}\} = \{1,\ 1\}$. For some randomly chosen real numbers, calculate the geometric mean $\left(\prod_{k=1}^{n} a_k\right)^{1/n}$ as a function of $n$. Does it exist?

**f)** The Brjuno function $\mathcal{B}(x)$ of an (irrational) real number $x$ (if it exists) is defined in the following way [349✱], [1234✱], [1316✱], [1317✱], [1318✱], [1232✱], [1172✱], [240✱], [192✱], [1212✱], [1233✱]:

$$\mathcal{B}(x) = \sum_{k=0}^{\infty} \beta_{k-1}(x) \ln\left(\frac{1}{\xi_k}\right)$$

$$\beta_{-1}(x) = 1$$
$$\beta_k(x) = |p_k(x) - x\, q_k(x)|$$

$$\xi_0 = x - \lfloor x \rfloor$$
$$\xi_k = \frac{p_k(x) - x\, q_k(x)}{x\, q_{k-1}(x) - p_{k-1}(x)}.$$

Here $p_k(x)$ and $q_k(x)$ are the numerators and denominators of the convergent fraction expansion of $x$. The Brjuno function $\mathcal{B}(x)$ obeys the following functional equation for $0 < x < 1$:

$$\mathcal{B}(x) = -\ln(x) + x\, \mathcal{B}\left(\frac{1}{x}\right).$$

Make a sketch of the graph of $\mathcal{B}(x)$ and check the functional equation for some random value of $x$ to at least 100 digits.

**g)** Let $p_n(x) / q_n(x)$ be the sequence of convergents of the continued fraction approximation of $x$. Define the function sum of errors function $\mathcal{P}(x)$ as [1541✱]

$$\mathcal{P}(x) = \sum_{k=0}^{\infty} \left(x - \frac{p_n(x)}{q_n(x)}\right)$$

where the sum might terminate for rational $x$. Visualize $P(x)$ and calculate a 20-digit approximation of the integral $\int_0^\infty P(x)\, dx$.

Carry out a numerical simulation demonstrating to at least three digits the following limit (valid for almost all $x$): $\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} \theta_n(x) = 1 / \ln(16)$ [260✱], [262✱]. Here $\theta_n(x)$ is the scaled error of the rational approximations of $x$, meaning $\theta_n(x) = q_n(x)^2\, |x - p_n(x) / q_n(x)|$.

**h)** The iteration

$$x_1 = \xi$$
$$x_{n+1} = \left(1 + \frac{1}{x_n}\right)^n$$

increases monotonic for increasing $n$ for exactly one positive $\xi = \tilde{\xi}$ [580✱]. Find this $\tilde{\xi}$ to 100 digits.

**i)** The square root-Bolyai expansion of a real number $x$ $(0 < x < 1)$ is of the form [1530✱], [263✱]

$$x = -1 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}\ .$$

The Bolyai digits $a_n \in \{0, 1, 2\}$ can be calculated via:

$$a_n = \alpha(\mathcal{T}^{n-1}(x))$$
$$\mathcal{T}(x) = (x+1)^2 - 1 - \alpha(x)$$

$$\alpha(x) = \begin{cases} 0 \text{ if } 0 \le x < \sqrt{2} - 1 \\ 1 \text{ if } \sqrt{2} - 1 \le x < \sqrt{3} - 1 \\ 2 \text{ if } \sqrt{3} - 1 \le x \le 1. \end{cases}$$

How many Bolyai digits are needed to approximate $1/\pi$ and $1/e$ to 1000 digits? Find exact values of the numbers that have the following repeating sequences of Bolyai digits: {1, 2}, {2, 1}, and {0, 1, 2}.

**j)** For a real number $x$ greater than 1, define the symmetric continued fraction expansion by

$$x = d_0(x) + \cfrac{d_0(x)}{d_1(x) + \cfrac{d_1(x)}{d_2(x) + \cfrac{d_2(x)}{d_3(x) + \cfrac{d_3(x)}{d_4(x) + \cfrac{d_4(x)}{d_5(x) + \cfrac{d_5(x)}{\cdots}}}}}}.$$

Here the $d_k(x)$ are positive integers. Implement a function `symmetricContinuedFraction` that calculates the first $o$ terms of the symmetric continued fraction expansion of a given number $x$. Calculate the first 100 terms of the symmetric continued fraction expansion of $\pi$, $e$, and $\sqrt{2}$. How precise are these approximations?

# CHAPTER 2

## Exercises

### 1.[L1] `DivisorSigma`, Primes, Maximum Formula, `StirlingS2`, `LegendreSymbol`, GCD-Iterations, Isenkrahe Algorithm, Multinomial Terms, Prime Divisors

**a)** The divisor function $\sigma_k(n)$ can be written in the form: If $n$ has the prime factorization (the $p_i$ are prime numbers) $n = \prod_{i=1}^{l} p_i^{\alpha_i}$, then

$$\sigma_k(n) = \prod_{i=1}^{l} \frac{p_i^{(\alpha_i+1)k} - 1}{p_i^k - 1}.$$

Implement this method of calculating $\sigma_k(n)$ in a one-liner.

**b)** Calculate the first six terms of the following sequence [613✱]:

$$p_{n+1} = \left\lfloor 1 - \log_2\left(\frac{1}{2} + \sum_{r=1}^{n} \sum_{1 \le i_1 < \cdots < i_r \le n} \frac{(-1)^r}{2^{p_{i_1} \cdots p_{i_r}} - 1}\right)\right\rfloor$$

$$p_1 = 2.$$

Here, $\lfloor x \rfloor$ denotes the integer part of $x$.

**c)** The maximum of $n$ integers (real numbers) $x_1, x_2, \ldots, x_n$ can be expressed through the minimum of these numbers in the following (inefficient) way [359✱]:

$$\max(x_1, x_2, \ldots, x_n) = \sum_{1 \le k_1 \le n} x_{k_1} - \sum_{1 \le k_1 < k_2 \le n} \min(x_{k_1}, x_{k_2}) + \sum_{1 \le k_1 < k_2 < k_3 \le n} \min(x_{k_1}, x_{k_2}, x_{k_3}) -$$
$$\ldots + (-1)^{j+1} \sum_{1 \le k_1 < k_2 < \ldots k_j \le n} \min(x_{k_1}, x_{k_2}, \ldots, x_{k_j}) +$$
$$\ldots + (-1)^{n+1} \sum_{1 \le k_1 < k_2 < \ldots k_{n-1} \le n} \min(x_{k_1}, x_{k_2}, \ldots, x_{k_n}).$$

Implement the above formula. How many calls to the function min are needed to calculate the maximum of the list {1, 2, 3, …13, , 14, 15}?

**d)** The Stirling numbers of the second kind $\mathcal{S}_n^{(m)}$ have the following neat representation [222✱], [607✱]:

$$\mathcal{S}_n^{(m)} = \sum_{1 \le k_1 \le k_2 \le \cdots \le k_{n-m} \le m} k_1\, k_2 \cdots k_{n-m}.$$

Here, $n \ge m$ is assumed. Implement this sum representation as a "one-liner".

**e)** In Section 2.2, we gave the following definition for the Legendre symbol $\left(\frac{a}{p}\right)$: Let $p$ be an odd prime. If $p$ divides $a$, $\left(\frac{a}{p}\right) = 0$. If this is not the case:

$$\left(\frac{a}{p}\right) = \begin{cases} +1 & \text{if } \exists\, x\text{, such that } x^2 = a \bmod p \\ -1 & \text{otherwise.} \end{cases}$$

Find an "explicit formula" for a function `LegendreSymbol[`*a*`, `*p*`]` that agrees with the built-in function `JacobiSymbol[`*a*`, `*p*`]` when $p$ is an odd prime and $a$ is a positive integer, and implement the formula as an "one-liner". Do not use boolean functions such as `PrimeQ`, procedural constructs such as `If`, quantifiers such as `Exists`, etc.

**f)** Analyze the number of steps need in carrying out Euclid's algorithm for determining $\gcd(p,\ q)$ for $1 \le p,\ q \le 256$.

**g)** The least common multiple of $r$ positive integers numbers $n_1,\ n_2,\ \ldots,\ n_k$ can be expressed through the minimum of these numbers in the following (inefficient) way [351✱], [505✱]:

$$\operatorname{lcm}(n_1, n_2, \ldots, n_r) = \frac{\prod_{i=1}^{r} \gcd(n_i) \prod_{i<j<k}^{r} \gcd(n_i, n_j, n_k) \cdots}{\prod_{i<j1}^{r} \gcd(n_i, n_j) \prod_{i<j<k<l}^{r} \gcd(n_i, n_j, n_k, n_l) \cdots}.$$

Implement the above formula. How many calls to the function `GCD` are needed to calculate the least common multiple of the list {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}?

**h)** Guess the limit of the following sequence [475✱], [14✱]:

$$\frac{1}{2},\ \frac{\frac{1}{2}}{\frac{3}{4}},\ \frac{\frac{\frac{1}{2}}{\frac{3}{4}}}{\frac{\frac{5}{6}}{\frac{7}{8}}},\ \ldots$$

**i)** Let $f_n(\xi)$ be the relative number of the sorted divisors $\{d_1, d_2, \ldots, d_{k(n)}\}$ of $n$ that are smaller or equal to $n^{\xi}$.

$$f_n(\xi) = \frac{l(n)}{k(n)}, \quad d_{l(n)} \le n^{\xi} < d_{l(n)+1}$$

Then, the averaged version of $f_n(\xi)$ (denoted by $F(\xi)$)

$$F_m(\xi) = \frac{1}{m} \sum_{n=2}^{m} f_n(\xi)$$

has asymptotically the form [241✱], [143✱]

$$F(\xi) = \lim_{m\to\infty} F_m(\xi) = \frac{2}{\pi} \arcsin\left(\sqrt{\xi}\right).$$

Use the first one million positive integers to see how well $F(\xi)$ is approximated.

**j)** Let $a^\dagger$, $a$ two noncommuting variables that obey the commutation relation $a\, a^\dagger - a^\dagger\, a = 1$. Consider the expansion

$$(a^\dagger\, a)^n = \sum_{j=0}^{n} c_{n,j}\, (a^\dagger)^j\, a^j.$$

Carry out this expansion for small $n$, and guess a closed form for the $c_{n,j}$.

**k)** Let $\lambda(p/q)$ be the length of the continued fraction expansion of the fraction $p/q$. For fixed denominator $n$, the cumulative average

$$l_q = \sum_{\substack{p=0 \\ \gcd(p,q)=1}}^{q} \lambda\left(\frac{p}{q}\right)$$

has asymptotically the form [449✱], [248✱], [570✱]

$$l_q \underset{q\to\infty}{\approx} \varphi(q) \left(\frac{12}{\pi^2} \ln(2) \ln(q) + c\right).$$

Calculate $l_q$ for $q \le 1000$ and $l_{10^6}$ and estimate the value of $c$.

**l)** The Isenkrahe algorithm [419✱] for the calculation for next prime $p$ after the prime $q$ is given as the fixed point of the map

$$n \longrightarrow \frac{n!}{\psi(q, n)} + \frac{\psi(q, n)}{(n-1)!} - \left\lfloor \frac{(n-1)!}{\psi(q, n)} \right\rfloor.$$

The function $\psi(m, n)$ (for fixed $n$) is defined as the following product over all primes $p$ less than or equal to $m$

$$\psi(m, n) = \prod_{p \le m} p^{\nu_p(n)}$$

$$\nu_p(n) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Implement the Isenkrahe algorithm as a one-liner.

**m)** The multinomial theorem describes the expansion of $\left(\sum_{j=1}^{m} a_j\right)^n$. It can be written in the following form [373✱]:

$$\left(\sum_{j=1}^{m} a_j\right)^n = \sum_{k=1}^{\max(m,n)} \Gamma_k^{(n)}(a_1, a_2, \ldots, a_m)$$

Here $\Gamma_k^{(n)}(a_1, a_2, \ldots, a_m)$ is the sum of all terms containing products of $k$ different $a_j$.

$$\Gamma_k^{(n)}(a_1, a_2, \ldots, a_m) = \sum_{\mu_1=1}^{m-(k-1)} \sum_{\mu_2=\mu_1+1}^{m-(k-2)} \cdots \sum_{\mu_k=\mu_{k-1}+1}^{m}$$

$$\sum_{\nu_1=1}^{n-(k-1)} \sum_{\nu_2=1}^{n-\nu_1-(k-2)} \cdots \sum_{\nu_{k-1}=1}^{n-\nu_1-\nu_2-\ldots-\nu_{k-2}-1} \sum_{\nu_k=n-\nu_1-\nu_2-\cdots-\nu_{k-1}}^{n-\nu_1-\nu_2-\cdots-\nu_{k-1}} (n; \nu_1, \nu_2, \ldots, \nu_m)\, a_{\mu_1}^{\nu_1}\, a_{\mu_2}^{\nu_2} \cdots a_{\mu_k}^{\nu_k}.$$

Implement the calculation of the $\Gamma_k^{(n)}(a_1, a_2, \ldots, a_m)$ and the expansion of $(\sum_{j=1}^m a_j)^n$ based on the $\Gamma_k^{(n)}$.

**n)** Visualize that on average, an integer $n$ has about $\omega(n) \propto \ln(\ln(n)) + B_1 + \gamma/\ln(n)$ different prime factors [62✱], [63✱], [562✱]. Here $B_1 = \sum_{k=2}^{\infty} k^{-1}\, \mu(k) \log(\zeta(k))$ is the Mertens constant.

**o)** The Fibonacci numbers $F_n$ with $n \geq 3$ can be characterized through the following property: an integer $k$ is a Fibonacci number if and only if the interval $[k\,\varphi - 1/\varphi, k\,\varphi + 1/\varphi]$ contains an integer [334✱]. Implement a recursive definition for the calculation of the Fibonacci numbers based on this property.

**p)** Consider the Kimberling sequence $a_k = (2^{-\nu_2(k)} + 1)/2$ [319✱], where $\nu_2(k)$ is the largest power of 2 that occurs as a factor in $k$ (in *Mathematica* `IntegerExponent[k, 2]`). The Kimberling sequence has the remarkable property that deleting the first occurrence of each positive integer in it results in the original sequence. Check this property for the first million terms of the Kimberling sequence.

**q)** Let $C(x)$ be the classical Cantor function. Is $\int_0^\infty C(x)^{C(x)}\, dx > 3/4$?

### 2.[L1] Cattle Problem of Archimedes

Solve the following cattle problem of Archimedes (see [252✱], [356✱]): The sun god Helios had a herd of cattle (steer and cows) that were either white, or black, or spotted, or brown. The number of white steer was equal to 1/2 + 1/3 the number of black steer more than the number of brown steer. The number of black steer was 1/4 + 1/5 the number of spotted steer more than the number of brown steer. The number of spotted steer was 1/6 + 1/7 the number of white steer more than the number of brown steer. The number of white cows was 1/3 + 1/4 the number of black cattle. The number of black cows was 1/4 + 1/5 the number of spotted cattle. The number of spotted cows was 1/5 + 1/6 the number of brown cattle, and the number of brown cows was 1/6 + 1/7 of the number of white cattle. How many of each type were in the herd of Helios?

### 3.[L1] Complicated Identity, $\pi$-Formula

**a)** Interactively check the following identity:

$$\sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} = \sqrt{22 + 2\sqrt{5}} - \sqrt{11 + 2\sqrt{29}} + \sqrt{5}$$

(from [133✱]; for a listing of related identities, see [133✱], [52✱], [53✱], and [350✱]).

**b)** Can a finite $n$ truncation of the following limit [384✱] be used for calculating $\pi$ to 10 digits? How long would it approximately take?

$$\pi = \lim_{n\to\infty} \sqrt{6\, \frac{\ln(F_1\, F_2 \cdots F_n)}{\ln(\text{lcm}(F_1, F_2, \ldots, F_n))}}$$

### 4.[L1] Mirror Charges

Consider the following electrostatic problem: Given two metal plates that meet at an angle of $\theta$, a point charge is located at an angle of $\varphi$ from the first metal plate. Here is a sketch.

Which of the following problems can be solved with the reflection charge method (see [293✱], [554✱], and [492✱]), and which cannot? Look at the solutions graphically, along with the failures of the construction:

$\theta = 90^\circ,\ \varphi = 45^\circ$
$\theta = 60^\circ,\ \varphi = 30^\circ$
$\theta = 90^\circ,\ \varphi = 22.5^\circ$
$\theta = 11.25^\circ,\ \varphi = 5.625^\circ$
$\theta = 20^\circ,\ \varphi = 5^\circ$
$\theta = 50^\circ,\ \varphi = 10^\circ$
$\theta = 120^\circ,\ \varphi = 40^\circ$
$\theta = 30^\circ,\ \varphi = 15^\circ$

### 5.[L1] Periodic Decimal Numbers, Digit Sequences, Numbered Permutations, Binomial Values, Smith's Sturmian Word Theorem

**a)** Built-in *Mathematica* functions can handle (exact) rational numbers and (approximate) real numbers, but not "exact real numbers with a decimal point". Write a routine `ExactDecimalNumber[`*fraction*`]` along the classical, school taught line of how to divide two integers writes the fraction in an exact way (possibly with a periodically repeating part). (The function `RealDigits` provides a similar functionality.)

**b)** Given a real number $x$ with decimal expansion $d_1\, d_2 \ldots d_k . d_{k+1} \ldots\ d_1 \neq 0$, form a sequence of integers $\{a_1, a_2, \ldots\}$ such that $a_k$ is the smallest integer formed by concatenating the consecutive $d_k$ that have not already occurred. For example, for $x = \pi = 3.1415926535897932384\ldots$, the corresponding sequence of integers is $\{3, 1, 4, 15, 9, 2, 6, 5, 35, 8, 97, 93, 23, 84, \ldots\}$. Implement a function `smallestIntegerSequence` that calculates the sequence $\{a_1, a_2, \ldots\}$ for a given real number $x$. What is the approximate growth rate of the $a_k$ for a random $x$?

**b)** The $n!$ permutations of a list of $n$ different elements $e_k$, $k = 1, \ldots, n$ can be conveniently numbered in the following way [161✱], [504✱], [425✱], [140✱]: The $k$th permutation is obtained by first writing $k$ in the form $k = \sum_{k=1}^{n} c_k (n-k)!$, where each $c_k$ is chosen nonnegative and as large as possible (factorial representation). Then, starting with the list $\ell = \{e_1, e_2, \ldots, e_n\}$, successively take the $(c_k + 1)$th element of the list, store it in a new list, and delete it from the original list. The list of the so-extracted list is the $k$th permutation. Implement a function `kthPermutation` which calculates the $k$th permutation of a given list.

**d)** Let $\mathcal{B}(o)$ be the integer counting the number of times the integer $m$ occurs as a binomial coefficient $\binom{n}{k}$ for nonnegative $n$ and $k$ [519✱], [1✱]. Implement a function that calculates the $\mathcal{B}(o)$ pairs $\{n_j, k_j\}$ such that $\binom{n_j}{k_j} = o$ for a given $o$. Find an $o$, such that $\mathcal{B}(o) = 8$.

**e)** Write a one-liner, that, for a given irrational number $\alpha$ $(0 < \alpha < 1)$ tests the first $n$ digits in Smith's Sturmian word theorem [525✱], [86✱], [369✱]. Smith's Sturmian word theorem says that the sequence of 0's and 1's defined through $\lfloor (k+1)\,\alpha \rfloor - \lfloor k\,\alpha \rfloor$, $k = 1, 2, \ldots$ is identical to the sequence $g = \lim_{k \to \infty} g_k$, where $g_0 = 0$, $g_1 = 0^{c_1 - 1} 1$, $g_k = g_{k-1}^{c_k} g_{k-2}$, where $g_j^m$ means $m$ repetitions of $g_j$, $c_k$ is the $k$th digits in the ordinary continued fraction expansion of $\alpha$ (not counting the leading 0), and concatenation of digits in the $g_k$ is implicitly understood.

### 6.[L2] Galton Board, Ehrenfest Urn Model, Ring Shift, Longest Common Subsequence, Riffle Shuffles

**a)** Model a Galton Board [189✱] and compare the resulting distribution with the expected theoretical distribution.

**b)** Let two dogs $A$ and $B$ share $n$ numbered fleas. Initially, $m$ fleas are on dog $A$ and $n-m$ fleas are on dog $B$. In each step, a random number $r$ between 1 and $n$ is chosen. As a result, the flea number $r$ switches its dog. Model how the number of fleas approaches their equilibrium distribution when initially 10000 fleas are on dog $A$. When initially 100 fleas are on dog $A$, carry out $10^7$ steps and analyze how often $k$ fleas are on dog $A$. Compare with the theoretical distribution for the equilibrium [16✱], [159✱], [393✱], [206✱], [207✱], [451✱].

**c)** Given a list of $n$ nonnegative integers, possible "moves" are

$$\{k_1, k_2, \ldots, k_j, k_{j+1}, \ldots, k_n\} \longrightarrow \{k_1, k_2, \ldots, k_j - 1, k_{j+1} + 1, \ldots, k_n\}$$

if $k_j \geq k_{j+1}$ and

$$\{k_1, k_2, \ldots, k_n\} \longrightarrow \{k_1 + 1, k_2, \ldots, k_n - 1\}$$

if $k_n \geq k_1$ [277✱], [214✱], [215✱].

For the starting list `{m, 0, …, 0}` of length $n$, for which $2 \leq m, n \leq 10$ does repeatedly carrying out moves leads to configurations so that no further moves are possible?

**d)** Consider the following sandpile model: Given a list of positive integers at each step, all elements that are greater than a given value $c$ are randomly split into two integers, and added to its neighbors [280✱]. Assuming periodic boundary conditions, implement this process and visualize the position of changed elements for various integer lists and values of $c$.

**e)** Given two list of random integers $\{s_1, s_2, \ldots, s_n\}$ and $\{t_1, t_2, \ldots, t_n\}$ $(1 \leq s_k, t_k \leq c)$, two (possibly empty) subsequences $\{s_{i_1}, s_{i_2}, \ldots, s_{i_l}\}$ and $\{t_{j_1}, t_{j_2}, \ldots, t_{j_l}\}$ with $s_{i_k} = t_{j_k}$ of identical elements can be found. The length of the longest possible common subsequence $l_{n,n}$ can be found by the following recursive definition [261✱], [92✱]:

$$l_{i,j} = \max(l_{i-1,j-1} + \delta_{s_i,t_j}, l_{i-1,j}\; l_{i,j-1})$$
$$l_{i,0} = l_{0,j} = 0.$$

For two random strings of length $n = 10^4$ and $c = 2$, calculate and visualize $l_{k,k} / k$ $(k = 1, \ldots, n)$.

**f)** The popular riffle shuffle of a deck of cards can be modeled in the following way [10✱], [41✱], [556✱], [187✱], [536✱], [119✱]: Split a deck of card into two piles. The probability that the two piles have $k$ and $n-k$ cards is $\binom{n}{k} 2^{-n}$. Then riffle the two piles together in such a way that the probability of taking a card from a given pile is proportional to the size of this pile.

A simple measure for the mixing-degree of a deck of cards is the number of rising subsequences contained in the deck [41✱], [536✱]. Without loss of generality, we assume the initial deck in the form {1, 2, …, $n$}. A rising subsequence of a list of integers is the maximal length subsequence of successive values. (For example, the sequence {8, 11, 1, 4, 3, 5, 10, 6, 7, 2, 9, 12} has the six rising subsequences {8, 9}, {11, 12}, {1, 2}, {4, 5, 6, 7}, {3}, and {10}.)

Carry out 1000 riffle shuffles for decks of length 52, 100, and 1000. Visualize the increase of the number of rising sequences of the repeatedly shuffled decks.

### 7.[L1] Friday the 13th and Easter

**a)** What is the probability that the 13th day of a month is a Friday? The days of the week (Sunday = 1, Monday = 2, Tuesday = 3, Wednesday = 4, Thursday = 5, Friday = 6, Saturday = 7) are given in our (Gregorian) calendar by a date of the form month/day/year, which can be computed by ([312✱], [48✱], [32✱], [249✱], [183✱], [363✱])

$$weekday = \left( \left\lfloor \frac{23\, month}{9} \right\rfloor + day + year + 4 + \left\lfloor \frac{z}{4} \right\rfloor - \left\lfloor \frac{z}{100} \right\rfloor + \left\lfloor \frac{z}{400} \right\rfloor \right) \bmod 7 - \delta$$

with

$$\delta = \begin{cases} 2 & month \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$z = \begin{cases} year - 1 & month < 3 \\ year + 0 & \text{otherwise.} \end{cases}$$

Here, $\lfloor x \rfloor$ is the largest integer contained in $x$.

**b)** What are the earliest and latest days (and the year in which they occur) for Easter? According to Gauss, Easter Sunday is the $i + j + 1$th day after the 21st of March, where

$$\begin{aligned}
a &= year \bmod 19 \\
b &= year \bmod 4 \\
c &= year \bmod 7 \\
d &= \left\lfloor \frac{(8[year/100] + 13)}{25} \right\rfloor - 2 \\
e &= \left\lfloor \frac{year}{100} \right\rfloor - \left\lfloor \frac{year}{400} \right\rfloor - 2 \\
f &= (15 + e - d) \bmod 30 \\
g &= (6 + e) \bmod 7 \\
h &= (f + 19\, a) \bmod 30 \\
i &= \begin{cases} 28 & h = 29 \\ 27 & h = 28 \text{ and } a \geq 11 \\ h & \text{otherwise} \end{cases} \\
j &= (2\, b + 4\, c + 6\, i + g) \bmod 7.
\end{aligned}$$

(See [548✱], [61✱], [376✱], [219✱], [362✱], [50✱], [565✱], and [251✱] and on-line at http://www.uni-bamberg.de/~ba1lw1/fkal.html). Again, $\lfloor x \rfloor$ is the largest integer contained in $x$.
For more on calendars, see also the package `Miscellaneous`Calendar`` of I. Vardi, [574✱] and [142✱].

### 8.[L1] Number of Lattice Points, Binomial Digits, Decreasing Partitions, Partition Moments

**a)** Determine the number $n$ and the distance (from the origin) of all grid points of a $d$-dimensional simple cubic lattice [503✱] inside of a sphere of given radius $r$ for moderate-sized radii, and plot the result $n(r)$ for various $d$.

**b)** Write a one-liner that, for a given positive integer $R$, counts the number $p(R)$ of relatively prime integers $m$ and $n$ such that $m^2 + n^2 \leq R^2$ [616✱]. Visualize $p(R)$ for $R \leq 1000$. Asymptotically we have $p(R) = 6/\pi\, R + O(R^{\alpha})$ for $R \to \infty$ [67✱], [339✱]. Estimate a value of $\alpha$ from the calculated data.

**c)** Analyze the mean and the fluctuations of the digits sum (in base 10) of $\binom{n}{\lfloor n/2 \rfloor}$ for $1 \leq n \leq 10000$. From this analysis, does one expect the fluctuations asymptotically to have a normal distribution?

**d)** Let $\lambda = \{n_1, n_2, \ldots, n_m\}$ be a strictly decreasing partition of the integer $\mu$, meaning $n_k > n_{k+1}$ and $\sum_{k=1}^{m} n_k = \mu$. For such a partition $\lambda$, let $\rho(\lambda, x) = \sum_{k=1}^{\infty} \theta(n_k - x)$. For $\mu = 100$, form the average of the $\rho(\lambda, x)$ over all partitions $\lambda$ of $\mu$ and compare this average with the function $g_\mu(x) = \ln(1 + \exp(-\sigma\, x))/\sigma$ where $\sigma = \pi/(2\,(3\,\mu)^{1/2})$ [609✱], [366✱], [578✱], [579✱].

**e)** Let $p_n(m)$ be the number of partitions of the positive integer $n$ into $m$ summands (ignoring the order of the summands). $p_n(m)$ obeys the recursion $p_n(m) = \sum_{j=1}^{\min(n-m,m)} p_{n-m}(j)$ [19✱].

For the moments $\mu_n^{(k)} = \sum_{j=1}^{m} m^k\, p_n(m)$, we have the following asymptotics for large $n$ [470✱], [269✱]:

$$\mu_n^{(k)} \underset{n\to\infty}{\sim} \frac{1}{4\sqrt{3}\,n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)\left(\frac{\sqrt{6n}}{\pi}\right)^k \Sigma_k\left(\ln\left(\frac{\sqrt{6n}}{\pi}\right)+\gamma,\ 1!\,\zeta(2),\ \ldots,\ (k-1)!\,\zeta(k)\right)$$

where $\zeta(z)$ is the Riemann Zeta function (in *Mathematica* `Zeta[z]`) and $\Sigma_n(x_1, x_2, \ldots, x_n)$ is

$$\Sigma_n(x_1, x_2, \ldots, x_n) = \sum_{j_1, j_2, \ldots, j_n} \frac{n!}{j_1!\, j_2! \ldots j_n!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \ldots \left(\frac{x_n}{n!}\right)^{j_n}$$

and the sum extends over all $n$-tuples $\{j_1, j_2, \ldots, j_n\}$, such that $\sum_{i=1}^{n} i\, j_i = n$. Compare the exact values of $\mu_n^{(k)}$ with the asymptotic values for $n = 1000$ and $k = 0, \ldots, 10$.

### 9.[L1] 15 and 6174

What do the following pieces of code compute?

**a)**

```
Fifteen[n_Integer?(# > 1&)] :=
   FixedPoint[Plus @@ Flatten[IntegerDigits /@ Divisors[#]]&, n]
```

**b)**

```
TwoOrThreeOrFourOrFiveOrSeven[n_Integer?(# > 1&)] :=
     FixedPoint[Plus @@ Flatten[IntegerDigits /@
        Flatten[Apply[Table[#1,{#2}]&, FactorInteger[#], {1}]]]&, n]
```

**c)**

```
f6174[n_Integer?(0 <= # <= 9999&)] :=
  Drop[FixedPointList[(#1.#2 - #1.Reverse[#2]& @@
     {10^Range[0, Length[#] - 1], #}&[
                         Sort[IntegerDigits[#, 10, 4]]])&, n], -1]
```

Generalize the process carried out by `f6174` an arbitrary base $b$.

### 10.[L3] Selberg Identity, Kluyver Identity, Goodwyn Property, Guiasu Formula, Prime Sums, Farey–Brocot Sequence, Divisor Sum Identities

**a)** Write a one-liner that proves the Selberg identity [20✱]

$$\Lambda(n)\ln n + \sum_{d|n} \Lambda(n)\,\Lambda\left(\frac{n}{d}\right) = \sum_{d|n} \mu(n)\ln^2\left(\frac{n}{d}\right)$$

for a given $n \geq 0$ (the sum goes over all divisors $d$ of $n$), where $\Lambda(n)$ is the Mangoldt function defined via

$$\Lambda(n) = \begin{cases} \ln n & n = p^m \text{ for some prime } p \\ 0 & \text{otherwise.} \end{cases}$$

for a given $n$ by direct calculation of both sides.

**b)** Show numerically for some positive integers $n$ that the so-called Kluyver identities [321✱] hold:

$$\sum_{\substack{\nu \\ \gcd(\nu,n)=1}} \cos(2\,\pi\nu\,/\,n) = \mu(n)$$

$$\sum_{\substack{\nu \\ \gcd(\nu,n)=1}} 2\,\sin(2\,\pi\nu\,/\,n) = e^{\Lambda(n)}$$

where $\mu(n)$, $\Lambda(n)$ are the Möbius function and the Mangoldt function (from part a).

**c)** Verify for some maximal denominator $n$ the following property: Every member (excluding the first and the last term) of the ordered Farey sequence of fractions less than 1 is equal to (see [146✱], [344✱], and [270✱]):

$$\frac{\text{numerator}(\mathit{leftNeighbor}) + \text{numerator}(\mathit{rightNeighbor})}{\text{denominator}(\mathit{leftNeighbor}) + \text{denominator}(\mathit{rightNeighbor})}$$

Make a picture of the Ford circles [179✱] and [47✱] of a Farey sequence. A Ford circle of the fraction $\frac{h}{k}$ is a circle with radius $1\,/\,(2\,k^2)$ and center $(h\,/\,k,\ 1\,/\,(2\,k^2))$.

**d)** The elements $\{\mathcal{F}_k^{(n)}\}_{k=0,\ldots,2^n}$ of the Farey–Brocot sequence $\mathcal{F}^{(n)}$ [368✱], [581✱], [203✱] of order $n$ are recursively defined as

$$\mathcal{F}_{2\,k+1}^{(n)} = \frac{\text{numerator}(\mathcal{F}_k^{(n)}) + \text{numerator}(\mathcal{F}_k^{(n)})}{\text{denominator}(\mathcal{F}_k^{(n)}) + \text{denominator}(\mathcal{F}_k^{(n)})},\ \ k = 0,\, 2,\, \ldots,\, 2^{n-1}$$
$$\mathcal{F}_{2\,k}^{(n)} = \mathcal{F}_k^{(n-1)},\ \ k = 0,\, 1,\, 2,\, \ldots,\, 2^{n-1}$$
$$\mathcal{F}_0^{(0)} = 0$$
$$\mathcal{F}_1^{(0)} = 1$$

This means that the sequence $\mathcal{F}^{(n+1)}$ is constructed from the sequence $\mathcal{F}^{(n)}$ by inserting mediants between all consecutive pairs and numbering (lower index) the resulting points consecutively. Given an interval $[a,\, b] \subset [0,\, 1]$, express this interval as the smallest union of disjoint intervals $\left[\mathcal{F}_{k_j}^{(j)},\, \mathcal{F}_{k_j+1}^{(j)}\right]$. Calculate the number of Farey–Brocot intervals $\left[\mathcal{F}_{k_j}^{(j)},\, \mathcal{F}_{k_j+1}^{(j)}\right]$ needed to cover the uniform subdivision of $[0,\, 1]$ in $2^{12}$ intervals of length $2^{-12}$.

**e)** Implement the following relation for the calculation of the number of primes [231✱], [232✱] less than $x$ in an efficient manner:

$$\pi(x) = -\sum_{k=1}^{\lfloor \log_2 x \rfloor} \mu(k) \sum_{n=2}^{\lfloor \sqrt[k]{x} \rfloor} \mu(n)\, \Omega(n) \left\lfloor \frac{\sqrt[k]{x}}{n} \right\rfloor.$$

Here $\lfloor x \rfloor$ is the integer part of $x$, $\mu(n)$ is the Möbius $\mu$ function, and $\Omega(n)$ is the number of prime factor of $n$. Is there a big advantage of an efficient implementation in comparison to the direct one?

**f)** The $n$th prime $p_n$ ($n$ even) can be represented as the addition and subtraction of all earlier primes in the form $p_n = \pm 1 \pm p_1 \pm \ldots \pm p_{n-1}$. Find such formulas for all primes $p_2,\ p_4 \ldots$ less than 100 [491✱], [146✱].

**g)** Define the functions [90✱]

$$Q_d(z) = \sum_{k=1}^{\infty} \sigma_d(k)\, z^k$$

where $\sigma_d(k)$ are the divisor sums. Visualize some $Q_d(z)$ as a function of the complex variable $z$.

For some small integers $\alpha_k$, $\beta_k$, and $\gamma_k$, relations of the following form exists:

$$\sum_{k=0}^{3} a_{\alpha_k} \, Q_{\alpha_k}(z) = \sum_{k=0}^{3} b_{\beta_k, \gamma_l} \, Q_{\beta_k}(z) \, Q_{\gamma_l}(z).$$

Use an "experimental mathematics" approach to find such relations.

### 11.[L2] Choquet Approximation, Magnus Expansion

**a)** Given two fractions $a_1 / b_1$ and $a_2 / b_2$, then, for the fraction $(a_1 + a_2)/(b_1 + b_2)$, we have $a_1 / b_1 < (a_1 + a_2)/(b_1 + b_2) < a_2 / b_2$ [176✱]. Use this relation to construct a rational approximation of `N[Pi, 50]`, `N[E, 50]`, `N[5^(1/3), 50]`, and `N[Sqrt[2], 50]`, starting with the nearest integers. Visualize how the upper and lower bounds move toward the given irrational number.

**b)** The solution of a system of linear, homogeneous differential equations

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{A}(t)\,\mathbf{x}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

where $\mathbf{x}(t) = \{x_1(t), \ldots, x_n(t)\}$ and $\mathbf{A}(t)$ is a $n \times n$ matrix with elements $a_{ij}(t)$ can be expressed as

$$\mathbf{x}(t) = e^{\mathbf{\Omega}(t)} . \mathbf{x}_0 .$$

The matrix $\mathbf{\Omega}(t)$ can be calculated in the following way (Magnus expansion [287✱], [378✱], [266✱], [589✱], [594✱], [65✱], [450✱], [408✱], [286✱], [168✱], [405✱], [163✱], [288✱], [167✱], [289✱]):

$$\mathbf{\Omega}(t) = \sum_{k=1}^{\infty} \mathbf{\Omega}_k(t)$$

$$\mathbf{\Omega}_k(t) = \sum_{j=0}^{n-1} \frac{B_j}{j!} \int_0^t S_n^{(j)}(\tau)\, d\tau$$

$$\mathbf{S}_n^{(0)} = \mathbf{A}(t)$$

$$S_n^{(j)} = \begin{cases} 0 & n > 1 \\ \sum_{m=1}^{n-j} (\mathbf{\Omega}_k(t).\mathbf{S}_{n-m}^{(j-1)} - \mathbf{S}_{n-m}^{(j-1)}.\mathbf{\Omega}_k(t)). \end{cases}$$

How many terms $\mathbf{\Omega}_k(t)$ are needed to obtain 20 correct digits for $\{x_1(1), x_2(1)\}$ for the solution of the system

$$\begin{aligned} x_1'(t) &= t\, x_1(t) + x_2(t) \\ x_2'(t) &= x_1(t) + t^2\, x_2(t) \end{aligned}$$

with initial conditions $x_1(0) = x_2(0) = 1$?

### 12.[L1] Rademacher Identity, Goldbach Problem, Optical Factoring

**a)** The so-called Rademacher identity ([455✱], [456✱], [452✱], [19✱]) for the partition function is (this is a generalization of the asymptotic expansion given in Section 2.3):

$$p(n) = \frac{1}{\sqrt{2}\,\pi} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\, \frac{d}{dn}\left(\frac{\exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

$$A_k(n) = \sum_{\substack{h=1\\ \gcd(h,k)=1}}^{n} \omega_{h,k} \exp\left(-\frac{2\pi i h n}{k}\right)$$

$$\omega_{h,k} = \exp\left(\pi i \sum_{m=1}^{k-1} \frac{m}{k}\left(\frac{h m}{k} - \left\lfloor \frac{h m}{k} \right\rfloor - \frac{1}{2}\right)\right).$$

Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

How many terms of the $k$ sum must be included to get, after rounding to the nearest integer, the correct result for $p(1000)$?

**b)** Write a one-liner that, for a given positive integer $n$, returns a list of all pairs of prime number indices $\{k_1, k_2\}$ such that $p_k + q_k = n$, where $p_{k_1}$, and $p_{k_2}$ are prime numbers and $(p_{k_1} \le p_{k_2})$, $k = 1, \ldots, m(n))$. Calculate all such pairs for $n = 10^6$. Can one calculate the number of prime pairs for $n = 10^9$? Compare the result with the asymptotic conjecture [242✱], [223✱], [587✱], [144✱], [129✱], [33✱], [169✱], [291✱]:

$$m(n) \underset{n\to\infty}{=} \left(\prod_{k=2}^{\infty} \frac{p_k(p_k-2)}{(p_k-1)^2}\right)\left(\prod_{k=2,\, p_k | n}^{\infty} \frac{p_k-1}{p_k-2}\right)\int_2^n \frac{1}{\ln(x)^2}\, dx.$$

**c)** Consider the sums $s_{n,m}(x) = \sum_{k=(1-n)/2}^{(n-1)/2} \exp(i\pi(k-x)^2/m)$ for odd $n$ and $m$. Conjecture a relation between the shape of the function $s_{n,m}(x)$ as a function of the real variable $x$ and the property that $m$ is a divisor of $n$.

### 13.[L1] Bernoulli and Fibonacci Numbers, Zeckendorf Representation

**a)** A number of identities involving Bernoulli numbers $B_n$ can be conveniently written in a symbolic way, where after expanding powers they are replaced by suffices $B^n \to B_n$ (see, for instance, [482✱], [460✱], [200✱], [77✱], [234✱], [463✱], [99✱], and [479✱] for details on this kind of symbolic rewriting). One of these identities is $(n > 1)$ (we use $\doteq$ to emphasize that this identity is a not a literal one): $B_n \doteq (B+1)^n$. Use this identity to implement the calculation of Bernoulli numbers.

In a similar way, the Bernoulli polynomial $B_n(x)$ can be represented in the following manner: $B_n(x) \doteq (B+x)^n$. Use this identity to implement the calculation of the first Bernoulli polynomials.

**b)** A number of identities involving Fibonacci numbers $F_n$ can be conveniently written in a symbolic way, where after expanding powers they are replaced by suffices $F^n \to F_{n+1}$, (see, for instance, [146✱]). One of these identities is $(n, m > 0)$ (we again use $\doteq$ to emphasize that this identity is a symbolic one):

$$F^{n+m} \doteq F^{n-m}(F+1)^m.$$

Check this identity for $0 \le n, m \le 12$.

**c)** Any nonnegative integer $n$ can be uniquely written as a sum of Fibonacci numbers (Zeckendorf representation) [614✱], [508✱]:

$$n = \sum_{k=1}^{s_n} F_{l_k^{(n)}}.$$

Here $l_1 \geq 2$ and $l_{k+1}^{(n)} \geq l_k^{(n)} + 2$. Implement a function `ZeckenDorfDigits` that for a given $n$ calculates the list of the $l_k^{(n)}$. Calculate `ZeckenDorfDigits[`$10^{1000}$`]`.

The function

$$\Sigma_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{k} (-1)^{s_l}$$

has the asymptotic form

$$\Sigma_n = G\left(\log_\phi(n)\right) + O\left(\frac{\sqrt{\ln(n)}}{n}\right)$$

where $G$ is a periodic function (period 1) [148✱]. Use the Zeckendorf representation of the first 100000 integers to visualize the function $G$.

Every real number $z$, $0 < z < 1$ can be uniquely written as (the Sylvester–Fibonacci expansion) [184✱]

$$z = \sum_{k=3}^{\infty} \frac{c_k}{F_k}$$

where $c_k \in \{0, 1\}$. Implement a function `SylvesterFibonacciDigits[`$z$`, `$n$`]` that calculates a list of the first $n$ nonvanishing "digits" $c_k$. Calculate `SylvesterFibonacciDigits[1/Pi, 100]`.

### 14.[L2] lcm–gcd Iteration, Generalized Multinomial Theorem, Ramanujan $\tau$ Function

**a)** Starting with a list of integers $\{l_1, l_2, \ldots, l_n\}$, one randomly selects a pair of integers $\{l_i, l_j\}$ from this list and replaces $l_i$ by $\mathrm{lcm}(l_i, l_j)$ and $l_j$ by $\gcd(l_i, l_j)$. Repeating this process with all pairs until a fixed point is reached leads uniquely to a set of integers, such that one number of every pair of integers from the resulting list divides the other. Implement the construction of this set of integers.

**b)** For a positive integer $n$, integers $p_k$, and $1 \leq k \leq m$, $m \geq 2$, Hurwitz gave the following generalization of the multinomial theorem [281✱], [474✱], [44✱], [615✱]:

$$A_n(x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m) == \sum_{k=0}^{n} \binom{n}{k} k!\,(k+x)\,A_{n-k}(x_1+k, x_2, \ldots, x_m, p_1-1, p_2, \ldots, p_m).$$

The functions $A_n(x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m)$ are defined via the formula

$$A_n(x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m) == \sum_{k_1, k_2, \ldots k_m = 0}^{n} (n; k_1, k_2, \ldots, k_m) \prod_{j=1}^{m} (x_j + k_j)^{k_j + p_j}.$$

Implement the calculation of the $A_n(x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m)$. Check the validity of the above identity for some values of $n$ and $p_1, p_2, \ldots, p_m$.

**c)** Let the Ramanujan $\tau$ function $\tau(n)$ be implicitly defined by

$$\sum_{k=0}^{\infty} \tau(n)\, z^n = z \prod_{k=1}^{\infty} (1 - z^k)^{24}.$$

$\tau(n)$ fulfills the following identity [21✱], [235✱]:

$$\tau(m)\,\tau(n) = \sum_{d|\gcd(m,n)} d^{11}\,\tau\left(\frac{m\,n}{d^2}\right).$$

Verify this identity for all $m$, $n$, and $m\,n \le 100$.

**d)** In [156✱], the following nice formula for the Ramanujan $\tau$ function was given:

$$\tau(n) = \sum_{\substack{a,b,c,d,e=-\infty \\ C}}^{\infty} \frac{(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)}{1!\,2!\,3!\,4!}.$$

The summation variables $a, b, c, d, e \in \mathbb{Z}$ have to fulfill the additional conditions $C$:

$$a+b+c+d+e=0$$
$$a^2+b^2+c^2+d^2+e^2 = 10\,n$$
$$\{a, b, c, d, e\} \bmod 5 = \{1, 2, 3, 4, 0\}.$$

Use this formula to calculate $\tau(1000)$. How many 5-tuples $\{a, b, c, d, e\}$ contribute to the sum? (The Ramanujan $\tau$ function is implemented in the package `` NumberTheory`Ramanujan` `` as `RamanujanTau`.)

The Ramanujan $\tau$ function can be expressed through divisor sums in the following form [400✱], [345✱]:

$$\tau(n) = \sum_k \beta_{c_k}\,\sigma_{c_k}(n) + \sum_{k,l} \beta_{c_k,c_l} \sum_{i=1}^{n-1} \sigma_{c_k}(i)\,\sigma_{c_l}(n-i).$$

Here the $\beta_{c_k}$, $\beta_{c_k,c_l}$ are integers and the $c_k$ are small integers. Find such a representation.

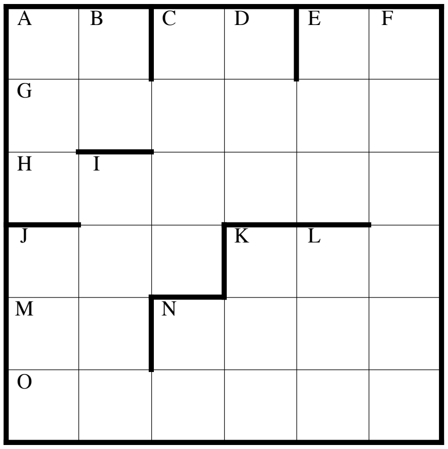
Still another (recursive) representation of the Ramanujan $\tau$ function is $\tau(n) = F_{n-1}(12, \tau(2), \ldots, \tau(n-1))$ [431✱]. Here the multivariate polynomials $F_n$ are defined by

$$F_n(x_1, x_2, \ldots, x_n) = -\frac{2}{n}\,x_1\,\sigma_1(n) + \sum_{\substack{m_1, m_2, \ldots, m_{n-1} \ge 0 \\ m_1 + 2 m_2 + \ldots (n-1) m_{n-1} = n}}^{\infty} (-1)^{\sum_{j=1}^{n-1} m_j} \frac{(-1+\sum_{j=1}^{n-1} m_j)!}{\prod_{j=1}^{n-1} (m_j!)} \prod_{j=1}^{n-1} x_{j+1}^{m_j}.$$

Use this formula to calculate $\tau(2), \ldots, \tau(25)$.

### 15.[L3] Cross-Number Puzzle

The *Berlin Intelligencer* (special issue of the Springer-Verlag publication *Mathematical Intelligencer* for the International Congress of Mathematicians, 1998; published in cooperation with the DMV) gave this cross-number puzzle:




*HORIZONTAL*
A) Prime number
B) The backnumber is a square
E) Prime number
G) The digits of this number are distinct
H) G)-horizontal - O)-horizontal
J) Square root of O)-horizontal
K) The cross-product is a square
M) Prime number
N) Multiple of E-vertical
O) The cross-sum is a square
*VERTICAL*
A) The backnumber divides G)-horizontal
B) A prime number to the power 4
C) The cross-sum equals C)-horizontal
D) M)-horizontal plus the backnumber of L)-vertical
E) The cross-product is prime
F) K)-horizontal times C)-vertical
I) Multiple of the backnumber of D)-vertical
J) The backnumber is a multiple of the backnumber of A)-vertical
K) A prime number to the power 5
L) Multiple of M)-horizontal
N) Prime number
The backnumber of a number is the number read backwards. Example: 5492 is the backnumber of 2945. The cross-sum is the sum of the digits and the cross-product is the product of the digits. There are no zeros.
Solve the cross-number puzzle using *Mathematica*.

### 16.[L2] Cyclotomic Polynomials, Generalized Bell Polynomials

**a)** The cyclotomic polynomials $C_n(z)$ ($n \in \mathbb{N}$) satisfy the following relations for prime $p$ [496✱]:

$$C_p(z) = \sum_{k=0}^{p-1} z^k$$

$$C_{n\,p^k}(z) = C_{n\,p}\left(z^{p^{k-1}}\right)$$

$$C_{n\,p}(z) = \frac{C_n(z^p)}{C_n(z)}$$

Together with $C_0(z) = 1$ and $C_1(z) = z - 1$, these relations completely define the cyclotomic polynomials. Based on the formulas given, implement the calculation of the first 100 cyclotomic polynomials.

**b)** The $n$th generalized Bell polynomial of order $m$, $Y_n^{(m)}(f_{1,1}, f_{2,1}, \ldots, f_{m+1,1}; f_{1,2}, f_{2,2}, \ldots, f_{m+1,2}; \ldots; f_{1,n}, f_{2,n}, \ldots, f_{m+1,n})$ with $n = 1, 2, \ldots, m = 2, 3, \ldots$ is recursively defined through [420✱]

$$Y_n^{(m)}(f_{1,1}, f_{2,1}, \ldots, f_{m+1,1}; f_{1,2}, f_{2,2}, \ldots, f_{m+1,2}; \ldots; f_{1,n}, f_{2,n}, \ldots, f_{m+1,n}) = \sum_{P(n)} \frac{n!}{r_1!\, r_2! \ldots r_n!}\, f_{1,r} \left(\frac{Y_1^{(m-1)}(f_{2,1}, \ldots, f_{m+1,1})}{1!}\right)^{r_1} \left(\frac{Y_2^{(m-1)}(f_{2,1}, \ldots, f_{m+1,1}; f_{2,2}, \ldots, f_{m+1,2}; \ldots; f_{1,n})}{2!}\right)^{r_2} \times \ldots \times \left(\frac{Y_n^{(m-1)}(f_{2,1}, \ldots, f_{m+1,1}; f_{2,2}, \ldots, f_{m+1,2}; \ldots; f_{2,n}, \ldots, f_{m+1,n})}{n!}\right)^{r_n}$$

with the initial term

$$Y_n^{(1)}(f_{1,1}, f_{2,1}; f_{1,2}, f_{2,2}; \ldots; f_{1,n}, f_{2,n}) = \sum_{P(n)} \frac{n!}{r_1!\, r_2! \ldots r_n!}\, f_{1,r} \left(\frac{f_{2,1}}{1!}\right)^{r_1} \left(\frac{f_{2,2}}{2!}\right)^{r_2} \times \ldots \times \left(\frac{f_{2,n}}{n!}\right)^{r_n}.$$

Here the sums extend over the set of all partitions $P(n) = \{\{r_1, r_2, \ldots, r_n\} \mid 1\, r_1 + 2\, r_2 + \ldots + n\, r_n = n\}$ of the integer $n$ and $r = r_1 + r_2 + \ldots + r_n$. Implement the calculation of the Bell polynomials using the given recursion relation.

The generalized Bell polynomials allow to express derivatives of nested functions via

$$\frac{d\, f_1(f_2(\ldots f_m(x)))}{d\, x^n} = Y_n^{(m-1)}\left(\tilde{f}_1', \tilde{f}_2', \ldots, \tilde{f}_m'; \tilde{f}_1'', \tilde{f}_{2,}'', \ldots, \tilde{f}_m''; \ldots; \tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}, \ldots, \tilde{f}_m^{(n)}\right)$$

where we use the abbreviation $\tilde{f}_j^{(k)} = f_j^{(k)}(f_{j+1}(f_{j+2}(\ldots f_m(x))))$. Check this relation for various $n$ and $m$.

### 17.[L2] Factorization of $n!$ into $n$ Factors, Bin Packing, Composition Multiplicities

**a)** Write a function that can construct all possible factorizations of $n!$ in exactly $n$ factors. This means a factorization of the form $n! = \prod_{j=1}^{n} p_j^{m_j}$, where all $p_j$ are prime numbers. For example, for $10! = 2^8\, 3^4\, 5^2\, 7$, we have

$10! = 2^1\, 2^1\, 2^1\, 2^1\, 2^1\, 2^1\, 2^2\, 3^4\, 5^2\, 7^1 = 2^1\, 2^1\, 2^1\, 2^1\, 2^1\, 2^3\, 3^1\, 3^3\, 5^2\, 7^1 = 2^1\, 2^1\, 2^1\, 2^1\, 2^1\, 2^3\, 3^2\, 3^2\, 5^2\, 7^1 = \ldots$ Try to avoid using the built-in function `FactorInteger`. How many possible factorizations of this kind exist for 25!?

**b)** Rods of length $1/3$, $2/3$, and 1 are randomly generated (each with probability $1/3$) and stuffed into boxes of length 1 (the so-called on-line bin packing problem [352✱]). The stuffing happens in such a way that, whenever possible, a new rod is put in a box that has still enough space and, if possible, a box is filled. After the generation of $n$ rods, the average cumulative rod length $\lambda_n$ in not completely filled boxes is [453✱], [322✱]

$$\lambda_n = \frac{1}{3^{n+1}} \left((2\,n+1)\binom{n,\,3}{n} + (2\,n+1)\binom{n-1,\,3}{n-1} + \binom{n-1,\,3}{n-2} - 3^n\right).$$

Here $\binom{n,\,3}{k}$ is the trinomial coefficient

$$\binom{n,\,3}{k} = [x^k]\,((1 + x + x^2)^n).$$

Model the stuffing process for $n = 1000$ by carrying out $10^4$ random realizations and compare the average cumulative rod length with the theoretical value. Model the generation of 10 trials for $n = 10^6$ rods and visualize how the average evolves as a function of $n$.

**c)** A composition of the positive integer $n$ into $p$ parts is a list of positive integers $\{k_1, k_2, \ldots, k_p\}$ that sum to $n$ [263✱], [264✱], [265✱], [256✱] (meaning a partition where the order of the summands matters). Write a program that, for a given integer $n$, forms all possible compositions. The multiplicity count of a composition $\lambda_j(\{k_1, k_2, \ldots, k_p\})$ is the number of $k_l$ having the value $j$. Calculate the cumulative multiplicity count for all compositions of $n = 50$ in a memory-efficient way.

### 18.[L2] Level Spacing, Subset Sums

**a)** The following function `levelSpacings` calculates the distribution of the differences of the sorted list of sums of two squares $k^2 + l^2 \leq n$.

```
levelSpacings[n_] := {First[#], Length[#]}& /@ Split[
 Sort[-Apply[Subtract, Partition[Sort[Flatten[
      Table[k^2 + l^2, {k, n}, {l, 1, Floor[Sqrt[n^2 - k^2]]}]]], 2, 1], {1}]]];
```

Rewrite this function in a form that uses less memory. Calculate `levelSpacings[1000]`, and visualize the result.

**b)** Choose $n$ random integers $k_1, k_2, \ldots, k_n$ from the interval $(1, N)$ and form all possible $2^n - 1$ sums of these integers containing each integer at most once (meaning $k_1, k_2, \ldots, k_n, k_1 + k_2, k_1 + k_3, \ldots, k_{n-1} + k_n, \ldots, k_1 + k_2 + \ldots + k_n$). Analyze the frequency of the values of the sums and compare with the theoretical distribution for large $n$ ($s$ is the value of the sum) [488✱], [489✱]

$$w(s) = \frac{1}{2^n - 1}\,\frac{\exp\left(\sum_{j=1}^{n} \ln(1 + e^{\beta k_j}) + \beta \sum_{j=1}^{n} k_j (1 + e^{\beta k_j})^{-1}\right)}{\sqrt{2\pi \sum_{j=1}^{n} k_j^2\,(1 + e^{\beta k_j})^{-1}\,(1 + e^{-\beta k_j})^{-1}}}$$

where $\beta$ is implicitly defined through $s = \sum_{j=1}^{n} k_j (1 + e^{\beta k_j})^{-1}$. Use $n \geq 20$ and $N \geq 100$.

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